特別寄稿

Turbulent Transport Measurements with a Laser Doppler Velocimeter

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1. Introduction

In a Laser Doppler Velocimeter (LDV) laser light is scattered from particles suspended in the fluid. The power spectrum of the scattered light contains information on the motion of the particles and hence information on molecular and turbulent transport processes. A schematic view of the measurement region is shown in Figure 1.

The light received at the detector is a linear sum of electric fields scattered from each particle in the illuminated region. The signal at the detector can be written

$$f(t) = A\cos(\omega_0 t + \phi(t)) \qquad (1)$$

where A is an amplitude, ω_0 is the laser frequency and $\phi(t)$ is the resultant phase.

$\phi(t)$ is the phase that results from properly adding the phase received from each of the particles. This phase is determined by the positions of the particles and the scattering angle, θ. For the situation shown in Figure 1, the phase of light scattered from the i^{th} particle is

$$\phi_i = \frac{2\pi\alpha}{\lambda} X_i + C \qquad (2)$$

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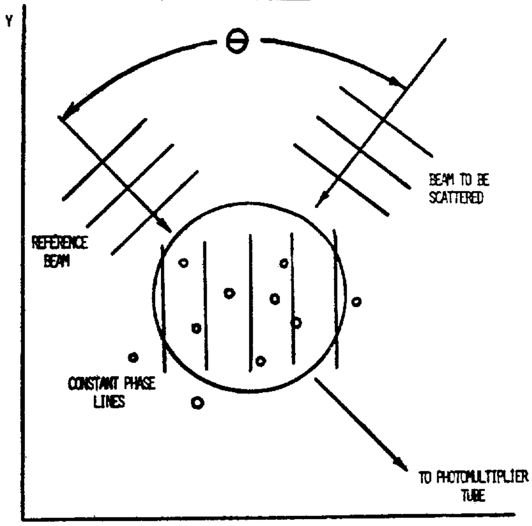

Fig. 1 Sample Region

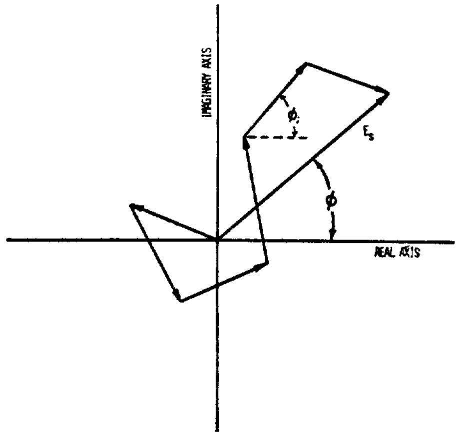

Fig. 2 Summation of Individual Fields

where X_i is the X coordinate of the i^{th} particle, λ the laser wavelength and α is a constant determined by the optics: $\alpha=2\sin(\theta/2)$. The resultant phase, ϕ, and amplitude, E_s, are obtained by summing the contributions, $E_i \exp i\ \phi_i$ from the individual illuminated particles. This is most easily seen graphically by a plot in the complex plane, Figure 2.

If the relative particle positions are fixed and if the suspending fluid is convecting at some constant velocity v in the X direction, the signal $f(t)$ is

$$f(t)=A(t)\cos[(\omega_0+\frac{2\pi\alpha}{\lambda}v)t+C] \qquad (3)$$

The term $\frac{2\pi\alpha}{\lambda}v$ is the frequency change usually interpreted as the Doppler shift. It is indeed proportional to the velocity and if this were all that happened, life would be simple indeed! However, as we mentioned, only the particles in the illuminated region contribute to the signal. If the particles in the illuminated region are removed and another set substituted by convection, the phase of the new signal is different, since the new particles do not occupy the same positions as the old set. Further, since the suspended particles are randomly placed in any real system, the new phase cannot be predicted solely with knowledge of the old phase; i.e., the two phase terms are uncorrelated, Therefore, in a real system with a finite illuminated volume, the signal (Eq. 3) only stays correlated with itself at time $t+\tau$ as long as some particles present in it at t are still present at time $t+\tau$. If the particles maintain their relative positions, this time is on the order of σ/v where σ is the characteristic dimension of the illuminated region in the flow direction.

The power density spectrum of a random signal is the Fourier transform of the autocorrelation of the signal. In this case, the autocorrelation is zero for all times greater than σ/v so that the resulting spectrum has a width on the order v/σ. To a first (but good) approximation this spectrum is centered at $2\pi\alpha\ \alpha/\pi$, so that a measurement of the spectrum can yield v and σ.

Until this point, the particles have been assumed not to change relative position and that the flow was steady.

If we now allow the particles to move relative to each other (diffuse), another phenomenon appears. In undergoing molecular or turbulent motion, the particles loss correlation with themselves, which means that the signal loses correlation with itself. This contributes to the spectrum of the received signal. The interpretation of this contribution of the spectrum in terms of the stochastic properties of turbulent fluid motion is the purpose of the remainder of this paper.

2. Theory

The instantaneous A.C. heterodyne current from the photodetector for a Laser Doppler Velocimeter may be written [1], assuming a constant reference beam E_r

$$i(t)\,\alpha E_s(t)E_r(t)=\sqrt{\text{const.}}\ Re\sum_n e^{ik\cdot(r_n(t))}P(r_n(t)), \quad (4)$$

where $r_n(t)$ is the time dependent position of the n^{th} particle in the system measured with respect to a laboratory frame. The vector K is the scattering vector and is fixed by the optics. $P(r_n(t))$ is the weighting function for the E fields seen by the photodetector. The weighting function, which can be complex, is determiend by the optics and can be computed from first principles. It is essentially the product of the E field amplitude weighting function for the two beams that define the sample volume. The term const. contains terms describing the scattering efficiency of the particles, the quantum efficiency of the photodetector, etc., which are unimportant for the present discussion.

By the Weiner-Khinchin theorem, the power density spectrum of the photodetector current can be written

$$I(K,\omega)=\int R_{ii}(K,\tau)\,e^{-i\omega\tau}\,d\tau, \qquad (5)$$

where

$$R_{ii}(K,\tau)=\text{const.}\ \overline{i(t)\,i(t+\tau)} \qquad (6)$$

is the autocorrelation of the current. The overbar denotes a time average.

This now may be written in terms of the particle positions,

$$R_{ii}=\text{const. } \overline{Re\left[\sum_{m}\sum_{n} e^{iK\cdot[r_n(o)-r_m(\tau)]}\, P(r_n(o))\, P(r_m(\tau))\right]} \tag{7}$$

or

$$R_{ii}=\text{const. } \Big[\overline{[Re[\sum_{n} e^{iK\cdot[r_n(o)-r_n(\tau)]}\, P(r_n(o))\, P(r_n(\tau))}+ \overline{2\sum_{n>m}\sum_{m}\cos\left(K\cdot[r_n(o)-r_m(\tau)]\right) P(r_n(o))\, P(r_m(\tau))}\Big] \tag{8}$$

Stationarity is assumed. In all but fluid-particle systems of very high number density, the relative particle positions can be taken as random. Therefore the second term in Eq. (8) is vanishingly small compared to the first term. Note here, that because of the average, the second term tends to zero even for the limiting case of very few particles in the sample volume at one time.

Equation 8 can now be rewritten,

$$R_{ii}=\text{const. } \overline{Re\sum_{n} e^{-iK\cdot\Delta\tilde{r}_n(\tau)} P(r_n(o))\, P(r_n(o)+\Delta\tilde{r}_n(\tau))} \tag{9}$$

where

$$\Delta\tilde{r}_n(\tau)=r_n(\tau)-r_n(o).$$

This expression can be formally evaluated in terms of two probability density functions:

1) that there is a particle within the volume element $r,\ r+dr$,

$$\frac{\rho}{N}dr; \tag{10}$$

2) that a particle in the volume element $r,\ r+dr$ moves a distance $\Delta\tilde{r}$, in time τ,

$$\tilde{G}(\Delta\tilde{r},\tau;r)\,d\Delta\tilde{r} \tag{11}$$

The autocorrelation of the photodetector current is from Equations 9, 10, and 11,

$$R_{ii}=\text{const. } Re\,\rho\iint e^{-iK\cdot\Delta\tilde{r}}\tilde{G}(\Delta\tilde{r},\tau;r)\,P(r)P(r+\Delta r(\tau))\,dr\,d\Delta\tilde{r} \tag{12}$$

3. Laminar Flow

Consider a system in steady, straight laminar flow. In this case, for the scattering particles normally used in LDV systems,

$$\tilde{G}(\Delta\tilde{r},\tau;r)=\delta(\Delta\tilde{r}-v(r)\tau) \tag{13}$$

i.e. the particles track the flow. Equation 12 becomes

$$R_{ii}=\text{const. } Re\,\rho\int e^{-ik\cdot v(r)\tau}P(r)\,P(r+v(r)\tau)\,dr \tag{14}$$

The dependence of v on r takes account of gradients in the velocity. Detailed calculations of the spectrum for laminar pipe flow have been presented in a previous paper [1]. A typical spectrum is shown in Figure 3. This spectrum demonstrates the effects of both the finite illuminated region and of a gradient in the velocity. Here v_0 is the velocity in the center of the sample volume and z_0/R is the fractional distance from the center of the pipe.

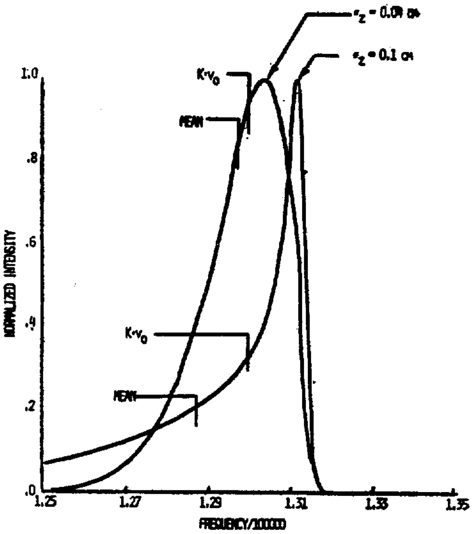

Fig. 3 Spectra observed with laminar pipe flow showing effects of large sample volume in radial (z) direction. In each case the sample volume was centered at 0.1cm from centerline of 1 cm pipe, the characteristic dimension, σ_x, in the flow direction was approximately .0045 cm and the velocity at the tube center was 10 cm/sec. Note that $K\cdot v_0$ is identical in each case.

4. Turbulent Flow

We define the average velocity $\bar{v}$ by

$$\bar{v}_{(r)}=\int\frac{\Delta\tilde{r}}{\tau}\tilde{G}(\Delta\tilde{r},\tau;r)\,d\Delta\tilde{r} \tag{15}$$

For convenience, $\Delta\tilde{r}$ will be defined in terms of the mean velocity $\bar{v}$, and Δr the deviation from the mean motion

$$\Delta\tilde{r}=\Delta r+\bar{v}\tau \tag{16}$$

The computations for this section will be done for $\bar{v}\neq\bar{v}\ (r)$. This is not a necessary condition, but it aids in the clarity of the presentation. Equation 12 now becomes

$$R_{ii}=\text{const. } Re\iint\tilde{G}(\Delta r+\bar{v}\tau)\,e^{-iK\cdot(\Delta r+\bar{v}\tau)K}\, P(r)\,P(r+\Delta r+\bar{v}\tau)\,dr\ d(\Delta r+\bar{v}\tau) \tag{17}$$

In view of Equation 15 we define

$$G(\Delta r,\tau)\,d\Delta r=\tilde{G}(\Delta\tilde{r},\tau)\,d\Delta\tilde{r} \tag{18}$$

where G is the probability that a particle moves Δr, $\Delta r+d\Delta r$ with respect to the mean motion $\bar{v}\tau$ in time τ. This definition becomes clear if one notes that the probability that a particle moves $\Delta\tilde{r}$ in time τ is identical to the probability that it moves Δr with respect to the mean flow in time τ.

Equation 17 becomes

$$R_{ii}=\text{const. } Re\int G(\Delta r,\tau)\,e^{-iK\cdot\Delta r,-iK\cdot\bar{v}\tau}Q_0(\Delta r+\bar{v}\tau)\,d\Delta r \tag{19}$$

where

$$Q_0=\int P(r)\,P(r+\Delta r+\bar{v}\tau)\,dr \tag{20}$$

Equation (19) can always be rewritten using the Fourier convolution and shift theorems as

$$R_{ii}=\text{const. } Re\int\hat{G}(K',\tau)\,e^{-iK'\cdot\bar{v}\tau}\hat{Q}_0(K-K')\,dK', \tag{21}$$

where $\hat{Q}_0$ is the Fourier space transform of Q_0 and $\hat{G}$ is the Fourier space transform of G.

R_{ii} can be shown to tend to zero with increasing τ for two reasons

1) $\lim_{\tau\to\infty}\left[\int e^{-iK'\cdot\bar{v}\tau}f(K')\,dK'\right]=0,$ (22)

for any square integrable $f(K')$,

2) $G(K',\tau)$ can be shown to be a decreasing function of τ for any K', in particular K, so that

$$\lim_{\tau\to\infty}\left[\int\hat{G}(K',\tau)\right)\hat{Q}_0(K'-K)\,dK']=0 \tag{23}$$

For the first mechanisum, we have demonstrated in a previous paper [1], that a characteristic decay time is

$$\tau_Q=2\sigma\sqrt{\bar{v}}$$

where σ is characteristic dimension of p^2 in the mean flow direction. For the second mechanism, the characteristic decay time can be taken to be when

$$\hat{G}(K,\tau)/\hat{G}(K,0)\approx e^{-\frac{1}{2}} \tag{24}$$

i.e. when the standard deviation $[\overline{\Delta K^2}_{(\tau)}]^{\frac{1}{2}}$, of $\hat{G}$ is approximately equal to K. But from the theory of Fourier transforms we know that

$$\overline{(\Delta K^2(\tau))}^{\frac{1}{2}}\,\overline{(\Delta r^2(\tau))}^{\frac{1}{2}}=0(1) \tag{25}$$

where $\overline{(\Delta r^2)}^{\frac{1}{2}}$ is the standard deviation of $G(\Delta,\tau)$.

The characteristic decay time, τ_0, for the second mechanism therefore occurs wehn

$$K[\overline{\Delta r^2}(\tau_0)]^{\frac{1}{2}}=0(1). \tag{26}$$

In other words, it is the time when the average scattering particle has diffused a distance $1/|K|$ with respect to the mean motion. Since $|K|$ for a typical LDV is on the order of $10^4\ \text{cm}^{-1}$, the second mechanism causes significant decay of R_{ii} when $\overline{(\Delta r^2)}^{\frac{1}{2}}\approx10^{-4}$ cm. This length is very small compared to the normally encountered Lagrangian integral length scales, Λ_L, which are on the order of the shear dimension of the flow system. Therefore, no matter what value τ_Q takes, R_{ii} decays in at most the time it takes the average particle to wander a distance with respect to the mean flow that is very small compared to the Lagrangian integral length scale Λ_L. This means that the integrand involved in the evaluation of R_{ii} is significantly different from zero only for times that are short compared to the Lagrangian integral time scale, τ_L.

In this case, $G(\Delta r, \tau)$ can be approximated by an asymptotic expression for small τ/τ_L.

$$G(\Delta r, \tau) \approx \int \delta(\Delta r - v'\tau)\, F(v';r)\, dv', \quad \approx \frac{1}{\tau} F\left(\frac{\Delta r}{\tau}; r\right) \tag{27}$$

where $F(v')$ is the probability density function for a turbulent fluctuation v'.

In this section, several simplifying assumptions are made:

1) $G(\Delta r, \tau)$ is isotropic
2) the flow is one dimensional
3) the vector K lies in the mean flow direction.

These assumptions are made only for clarity of presentation and do not represent a restriction on the method of calculating the spectra.

If the $G(\Delta r, \tau)$ from equation 27 is substituted into equation 19, and that used in equation 5, we find after some rearrangement

$$I(K, \omega) = \text{const.} \int \frac{1}{v'+\bar{v}} F(v')\, \hat{Q}_0\left(\frac{K\bar{v}-\omega+Kv'}{v'+\bar{v}}\right) dv' \tag{28}$$

where $v' = \frac{\omega}{K'} - \bar{v}$. The integration variable, v', in equation 28 can now be interpreted as the fluctuating velocity. This becomes clear when one recognizes that the term

$$\frac{1}{\bar{v}+v'} \hat{Q}_0\left(\frac{K(\bar{v}+v')-\omega}{\bar{v}+v'}\right) \tag{29}$$

is the spectrum generated by a flow with a constant velocity $\bar{v}+v'$. (See reference 1) Equation 28 gives the spectrum as a weighted sum of spectra generated by each velocity present in the sample region. Individual particles can change their velocities during their traverse of the sample volume; but since $\tau_0 \ll \tau_L$ they do not change their velocity significantly over, τ_0, the decay time for R_{ii}.

For $P=1$, a very large sample volume,

$$I(K, \omega) = \text{const.} \frac{1}{K} F\left(\frac{\omega - K\cdot\bar{v}}{K}\right) \tag{30}$$

In this case the spectrum has the shape of the probability density function for the fluctuating velocity with a scale factor of K.

If K is sufficiently small, $\hat{G}(K, \tau)$ and hence R_{ii} for turbulent flow, do not decay until times long compared to τ_L. In this case the observed particle motion takes on a Brownian motion character and $G(\Delta r, \tau)$ can be approximated by (4)

$$G(\Delta r, \tau) \approx \frac{1}{(4\pi\epsilon\tau)^{3/2}} e^{-\frac{\Delta r^2}{4\epsilon\tau}}, \tag{31}$$

where ε is the eddy diffusivity. Using equations 31, 21 and 5, the spectrum of the LDV becomes

$$I(K, \omega) = \text{const.} \int \frac{K'^2\varepsilon}{(K'^2\varepsilon)^2 + (\omega - K'\cdot\bar{v})^2} \hat{Q}_0(K'-K)\, dK'. \tag{32}$$

For equation 32 to be valid K must be less than $1/\Lambda_L$. Since Λ_L is typically greater than 1 cm., it appears infeasible to measure eddy diffusivities by the above technique. However the theory presented here also applies to ultrasonic scattering where small K vectors are possible. A check of the small K limit predicted here is feasible by ultrasonic techniques. For the Brownian motion of small particles, where the correlation length is much smaller than $1/K$, the results have been confirmed. [2, 3]

5. Calculation for Real Systems

Consider a three dimensional system with a sample volume having a Gaussian intensity with RMS widths $\sigma_x, \sigma_y, \sigma_z$ in the three orthogonal directions and with orthogonal turbulence intensities,

$$\frac{(\overline{v_x'^2})^{\frac{1}{2}}}{\bar{v}},\ \frac{(\overline{v_y'^2})^{\frac{1}{2}}}{\bar{v}},\ \frac{(\overline{v_z'^2})^{\frac{1}{2}}}{\bar{v}}.$$

the moments of the spectrum can be calculated from the three dimensional form of equation 28. When the mean flow is in the X-direction, and when K is in the X-direction, the mean and second central moment of the spectrum are

$$M_1=K\bar{v}, \tag{33a}$$

$$\mathrm{VAR}=\overline{Kv_x'^2}+\frac{\bar{v}^2}{4\sigma_x^2}+\frac{\overline{v'^2_x}}{4\sigma_x^2}+\frac{\overline{v'^2_y}}{4\sigma_y^2}+\frac{\overline{v'^2_z}}{4\sigma_z^2} \tag{33b}$$

or

$$\mathrm{VAR}=K^2\overline{v_x'^2}+\frac{\bar{v}^2}{4\sigma_x^2}\Big[1+\frac{\overline{v'^2_x}}{\bar{v}^2}+\frac{\overline{v'^2_y}}{\bar{v}^2}\Big(\frac{\sigma_x^2}{\sigma_y^2}\Big)+\frac{\overline{v'^2_z}}{\bar{v}^2}\Big(\frac{\sigma_x^2}{\sigma_z^2}\Big)\Big] \tag{33c}$$

These formulae are independent of the particular form of the fluctuating velocity probability density $F(v')$. The $\bar{v}^2/4\sigma^2$ terms have been discussed in ref. [1] and are the finite sample volume contribution to the width of the spectrum. The other terms are the broadening due to the turbulence. One will notice that variance is not simply the sum of the square of the finite volume broadening and the turbuulence broadening. It contains, in addition, turbulent finite volume broadening (e. g. $\overline{v_x'^2}/4\sigma_x^2$) due to the RMS velocity fluctuations carrying the scattering centers across the sample volume. Typically, $\sigma_x/\sigma_z<\sigma_x/\sigma_y\approx1$ and $(\overline{v'^2}/\bar{v}^2)^{\frac{1}{2}}\leqslant0.05$, so the effect of y and z fluctuations is to increase the finite transit time term by less than 3%. Since, in a well-designed experiment, the finite transit time broadening is small compared to the turbulence broadening, the effect of y and z fluctuations is negligible. Therefore, a good approximation to the variance of the spectrum is

$$\mathrm{VAR}=K^2\overline{v'^2_x}+\frac{\bar{v}^2}{4\sigma_x^2} \tag{34}$$

If the K-vector is in the y direction,

$$M_1=0, \tag{35}$$

$$\mathrm{VAR}=K^2\overline{v'^2_y}+\frac{\bar{v}^2}{4\sigma_x^2} \tag{36}$$

The LDV is therefore capable of measuring each component of the fluctuating intensity independently.

Near the center line of turbulent channel flow, the gradient in the mean velocity is small. Further $F(v')$ can be approximated by

$$F(v')\,dv'=\frac{1}{\sqrt[3]{2\pi\overline{v'^2_x}\ \overline{v'^2_y}\ \overline{v'^2_z}}}\exp-\Big[\frac{v'^2_x}{2\overline{v'^2_x}}+\frac{v'^2_y}{2\overline{v'^2_y}}+\frac{v'^2_z}{2\overline{v'^2_z}}\Big]dv'_xdv'_ydv'_z \tag{37}$$

Using equations 37 and 28

$$I(K,\omega)\approx\frac{\mathrm{const.}}{\sqrt{2\pi\overline{v'^2_x}}}\int_{-\infty}^{\infty}e^{-\frac{v'^2_x}{2\overline{v'^2_x}}}\frac{\exp}{(\bar{v}_x+v'_x)}-\Big[\frac{2\sigma_x^2\,(K(\bar{v}_x+v'_x)-\omega)^2}{(\bar{v}_x+v'_x)^2}\Big]dv'_x \tag{38}$$

Sine $|\bar{v}_x|\gg|v'_x|$ one has

$$\approx\frac{\mathrm{const.}}{\sqrt{2\pi\overline{v'^2_x}}}\int_{-\infty}^{\infty}\frac{1}{\bar{v}_x}\exp-\Big[\frac{v'^2_x}{2\overline{v'^2_x}}\Big]\exp$$

$$\Big[-\frac{2\sigma_x^2\,(Kv'_x+K\bar{v}_x-\omega)^2}{\bar{v}_x^2}\Big]dv'_x \tag{38b}$$

$I(K,\omega)\approx$

$$\frac{\mathrm{const.}}{\sqrt{2\pi\,[\bar{v}_x^2+2K^2\overline{v'^2}\sigma_x^2]}\ 2}\exp-\Big[\frac{(\omega-K\bar{v}_x)^2}{2\,(\frac{\bar{v}_x^2}{4\sigma_x^2}+K^2\overline{v'^2_x})}\Big] \tag{39}$$

The heterodyne turbulence spectrum in this case is approximated by a Gaussian with variance $(\frac{\bar{v}_x^2}{4\sigma^2_x}+K^2\overline{v'^2})$. Figure 4 is a turbulence spectrum taken channel center. The solid line is a Gaussian fitted to the data.

Near the center of a channel, the dependence of the turbulence intensity on position is not strong, so that over the sample volume, $\overline{v'^2}$, can be considered constant and the gradient in the mean velocity can be handled as in ref. [1]. Near walls where large gradients in the mean velocity may be present and a strong positional dependenc of $\overline{v'^2}$ may be also present, a more complex model for $G(\Delta r, \tau)$ must be used. This will be treated in a later paper.

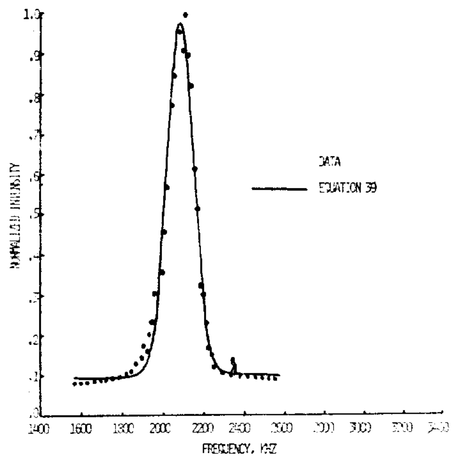

Fig. 4 Spectrum obtained from turbulent flow of water. Parameters were: $\bar{v}$=288 cm/sec, Re=85,700, θ=19.8°, σ_x=10.4 μm. Calculated percent turbulence was 2.95 %.

6. Conclusion

The power spectrum of phototube current for a Laser Doppler Velocimeter (LDV) operating in the heterodyne mode has been computed for the case of turbulent flows. It is shown that for normal operating parameters the spectrum contains information only on the short time behavior of the fluid motion. To examine the long time behavior, i.e., times greater than the Lagrangian integral time scale of the turbulence, one must use extremely small scattering angles.

It should be noted that this paper describes the power spectrum of a signal averaged over long times. This includes situations in which there may be a very few or very many particles in the sample volume at one time. We have not considered the case of F.M. detection. With F.M. detection it is possible to extend the time and length scale of the LDV measurement beyond those indicated in this paper.

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