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The Onset of Tayler-Görtler Vortices in Impulsively Decelerating Circular Flow

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Abstract – The onset of instability induced by impulsive spin-down of the rigid-body flow placed in the gap between two coaxial cylinders is analyzed by using the energy method. In the present stability analysis the growth rate of the kinetic energy of the base state and also that of disturbances are taken into consideration. In the present system the primary flow is a transient, laminar one. But for the Reynolds number equal or larger than a certain one, *i.e.* $Re \geq Re_G$ secondary motion sets in, starting at a certain time. For $R_e \geq Re_G$ the dimensionless critical time to mark the onset of vortex instabilities, τ_c, is here presented as a function of the Reynolds number Re and the radius ratio η. For the wide gap case of small η, the transient instability is possible in the range of $Re_G \leq Re \leq Re_S$. It is found that the predicted τ_c-value is much smaller than experimental detection time of first observable secondary motion. It seems evident that small disturbances initiated at τ_c require some growth period until they are detected experimentally.

Key words: Taylor-Görtler Vortex, Energy Method, Relative Stability

1. Introduction

In flows along concavely curved walls, the destabilizing action of centrifugal forces can produce an instability motion in form of stationary vortices. This instability is analogous to that of Taylor-Görtler vortices. The impulsive spin-down of initial rigid-body flow between two coaxial cylinders evolves into a secondary flow pattern which consists of a series of Taylor-like vortices. In this transient boundary-layer system the critical time t_c to mark the onset of secondary motion becomes an important question. In this connection the instability problem of decelerating circular flow has attracted interests.

Tillman[1] first investigated experimentally the onset of instability in the flow system of spin-down from solid-body rotation of coaxial cylinders filled with liquid suddenly brought to rest. The analytical difficulties involved in the application of conventional stability theory to this kind of transient flow has been considered[2] and the related instability analysis has been conducted by using the strong and the marginal stability criteria[3-5]. The strong stability criterion pursued the stability bounds in terms of time interval where the kinetic energy of disturbances starts to increase. The marginal stability criterion which relaxes the strong one shifts the stability bounds to a more stable direction. However it has been faced with mathematical difficulties. These models consider some finite, initial disturbances and trace the temporal growth of their kinetic energy.

In the present study, we will analyze the onset of Taylor-Görtler vortices in impulsively decelerating transient circular flow between coaxial two cylinders. This problem was already analyzed using the aforementioned models. We will relax the strong stability criterion by introducing the relative one, which has been used in the various problems[6-10]. The new stability equations will be derived for the whole time region and the resulting predictions will be compared with available experimental and theoretical results. Also, the effects of stability criteria on the critical conditions will be examined. Since the present system is a rather simple one, the present results will be helpful for comparison among available models.

2. Theoretical Analysis

2-1. Governing Equations

The system considered here is a Newtonian fluid confined between the cylinders of radii R_i and R_o. Let the axis of the cylinders be along the vertical z′-axis under the cylindrical coordinates (r′, θ, z′) and the corresponding velocities be *U*, *V* and *W*. The entire fluid/cylinder system is assumed to be in a state of rigid-body rotation with angular velocity Ω. Starting from time $t = 0$, the outer cylinder is impulsively stopped. The ensuing unsteady flow is known as spin-decay one. The schematic diagram of the present system is shown in Fig. 1. Such transient circular flow is known to be subjected to instability in form of Taylor-Görtler vortices and the governing equations of the flow field are expressed as

$$\nabla \cdot \mathbf{U} = 0 , \tag{1}$$

$$\left\{\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right\}\mathbf{U} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{U} , \tag{2}$$

where **U**, P, ν and ρ represent the velocity vector, the dynamic pressure, the kinematic viscosity and the density, respectively.

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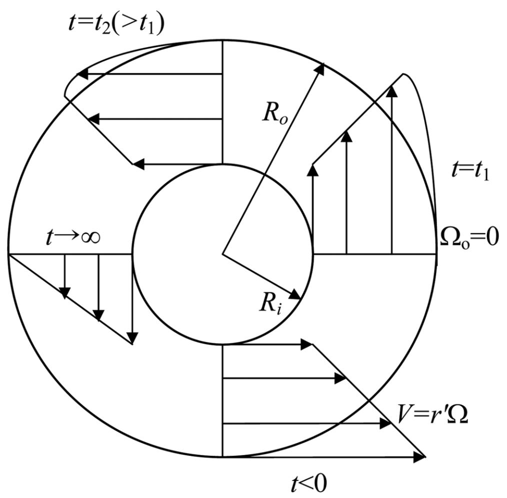

Fig. 1. Top view of the system considered here.

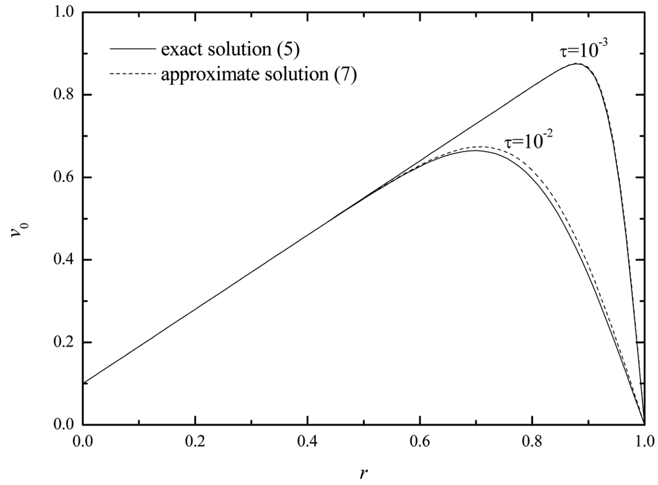

Fig. 2. Primary-velocity profiles for η = 0.1.

The primary-velocity field is represented for the case of constant physical properties:

$$\frac{\partial V_0}{\partial t} = \nu D' D'_* V_0 \,, \tag{3}$$

with the following initial and boundary conditions,

$$V_0(0, r') = r'\Omega \,,\; V_0(t, R_i) = R_i\Omega \text{ and } V_0(t, R_o) = 0. \tag{4}$$

where $D' = \partial/\partial r'$ and $D'_* = D' + 1/r'$. Neitzel [4] obtained the following analytical, exact solution as

$$v_0(\tau, r) = \frac{V_0}{R_o\Omega} = -\frac{\eta^2}{(1+\eta)}\left(r - \frac{1}{(1-\eta)^2}\right)$$

$$+(1-\eta)\sum_{i=1}^{\infty} k_n[J_1(\lambda_i r) + c_i Y_1(\lambda_i r)]\exp(-\lambda_i^2\tau) \,, \tag{5a}$$

where

$$k_n = 2Z_0\left(\frac{\lambda_i}{1-\eta}\right)\left\{\lambda_i\left[\eta^2 Z_0^2\left(\frac{\lambda_i\eta}{1-\eta}\right) - Z_0^2\left(\frac{\lambda_i}{1-\eta}\right)\right]\right\}^{-1}. \tag{5b}$$

In the above J and Y are Bessel functions of the first and the second kind, respectively, $r = (r' - R_i)/d$, $\eta = R_i/R_o$ and $\tau = \nu t/d^2$, here $d = R_o - R_i$. (λ_i, c_i) are the roots of the equations

$$Z_1(\lambda_i\eta/(1-\eta)) = 0 \tag{5c}$$

and

$$Z_1(\lambda_i/(1-\eta)) = 0 \tag{5b}$$

where

$$Z_k \equiv J_k + c_i Y_k \tag{5d}$$

For the limiting case of $\eta\to 1$, *i.e.* very narrow gap, the curvature effects can be negligible and the above velocity profile can be represented by using the complementary error function as

$$v_0(\tau, r) = 1 - \sum_{n=0}^{\infty}\left\{\operatorname{erfc}\left(\frac{n}{\sqrt{\tau}} + \frac{1-r}{2\sqrt{\tau}}\right) - \operatorname{erfc}\left(\frac{n+1}{\sqrt{\tau}} - \frac{1-r}{2\sqrt{\tau}}\right)\right\}$$

$$\text{for } \eta\to 1. \tag{6}$$

For small time, the velocity profiles of Eqs. (5) and (6) can be approximated as

$$v_0(\tau, r) = 1 - (1-r)(1-\eta) - \operatorname{erfc}\frac{1-r}{2\sqrt{\tau}} \quad \text{for } \tau\to 0. \tag{7}$$

The instantaneous base flow profile is shown in Fig. 2. As shown in this figure for $\tau \le 10^{-3}$, the deep-pool solution (7) approximates the exact solution (5) quite well. To reduce computation time, Eq. (7) is used in stability analysis for the region of $\tau \le 10^{-3}$. From this profile the centrifugal instability near the outer cylinder wall can be expected based on the Rayleigh criterion for the inviscid flow[5]. However, sophisticated stability analysis is required to obtain stability limit since present system is time-dependent and viscous.

2-2. Energy Method

Following the work of Serrin[11] and Neitzel[4], the energy identity is written as

$$\frac{dE}{d\tau} = \mathrm{Re}I - D \,, \tag{8}$$

where $E = \langle \mathbf{u}\cdot\mathbf{u}\rangle/2$, $I = \langle u v \phi\rangle$, $\phi = (1/r - \partial/\partial r)v_0$ and $D = \langle\nabla\mathbf{u}:\nabla\mathbf{u}\rangle$. Here $\mathbf{u}(=\mathbf{U}/V_i)$ is the dimensionless velocity vector, $\mathrm{Re}(=V_i d/\nu)$ is the Reynolds number and $\langle\cdot\rangle$ represents the average over the system. The conventional energy method determines the critical times to mark the onset of secondary motion at which E is the minimum, *i.e.*

$$\frac{dE}{d\tau} = 0 \text{ at } \tau = \tau_s. \tag{9}$$

This condition is known as the strong stability criterion[2]. Neitzel[4] relaxed this strong stability criterion by considering the growth of the disturbance kinetic energy. For a given Re, the marginal stability τ_m is determined implicitly from the condition of

$$\int_0^{\tau_m}\sigma(\tau')d\tau' = 0 \,, \tag{10}$$

where the growth rate is defined as

$$\sigma(\tau) = \max[(\mathrm{Re}I - D)/E]\,. \tag{11}$$

The time τ_m means the fastest time for the disturbance kinetic energy to recover its initial value, *i.e.*

$$E(\tau) = E(0) \text{ at } \tau = \tau_m\,. \tag{12}$$

As discussed by Gummerman and Homsy[12] and Neitzel[4] the growth rate $\sigma(\tau)$ cannot be obtained explicitly and therefore, the calculation of τ_m suffers from serious computational burden. Owing to this kind of difficulties Gummerman and Homsy[12] and Neitzel[4] obtained the stability limit for the limited domain.

Here, we relax the above stability limits by introducing the relative stability concept: the temporal growth rate of the kinetic energy of the disturbance velocity should exceed that of the base velocity at the onset condition of secondary motion. This stability criterion was proposed by Chen et al.[2], and applied into the various problems by Kim et al.[6-10]. In the relative stability model the critical time τ_r is determined, based on a most dangerous mode of instability:

$$\sigma_1 = \sigma_0 \text{ at } \tau = \tau_r\,, \tag{13}$$

where $\sigma_1 = (1/E)(dE/d\tau)$, $\sigma_0 = (1/E_0)(dE_0/d\tau)\,d\tau$ and E_0 is the basic centrifugal potential energy, *i.e.* $E_0 = \langle\{v_0(\tau, r) - v_0(0, r)\}^2\rangle/2$. The above criterion means that secondary motion sets in at τ_r when the growth rates of the energy of disturbance and base quantity are the same. In the strong and marginal stability criterion, only the decay or growth of disturbance quantity is taken into account. Based on Eqs. (8) and (13), the relaxed energy identity for the relative stability model becomes

$$\sigma_0 E = \mathrm{Re}I - D\,. \tag{14}$$

Now, the relative stability limit is given by

$$\frac{1}{\mathrm{Re}} = \max\left[\frac{I}{D + \sigma_0 E}\right]. \tag{15}$$

Under the normal mode analysis the typical axisymmetric disturbances, which have been observed experimentally[1,13,14] and known as the energetically most unstable mode[5], are well represented by

$$(u_1, v_1, p_1) = (u', v', p')\cos az\,, \tag{16a}$$

$$w_1 = w'\sin az, \tag{16b}$$

where a is the dimensionless wavenumber representing the periodicity in the z′-direction, z = z′/d and the primed quantities representing disturbance amplitudes are a function of r and τ. Here we assume the infinitely long cylinder and neglect the endwalls effects. Then this maximum problem can be solved by the variational technique[11]. By eliminating the Lagrage multiplier term with the aid of continuity equation, the Euler-Lagrange equations for the relative stability model are obtained:

$$-\frac{1}{2}a^2\mathrm{Re}\phi v' + \left\{\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) - a^2\right\}^2 u' = \frac{\sigma_0}{2}\frac{\partial^2 u'}{\partial r^2}\,, \tag{17}$$

$$\frac{1}{2}\mathrm{Re}\phi u' + \left\{\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) - a^2\right\} v' = \frac{\sigma_0}{2}v'\,. \tag{18}$$

The proper boundary conditions are

$$u' = \frac{\partial u'}{\partial r} = v' = 0 \text{ at } r = 0 \text{ and } 1. \tag{19}$$

Based on the velocity profile of Eq. (5), the growth rate of basic kinetic energy is given as

$$\sigma_0 = \frac{-2\lambda_i^2\sum_{i=0}^{\infty}\exp(-\lambda_i^2\tau)\{b_i\exp(-\lambda_i^2\tau) + c_i\}}{f(\eta) + \sum_{i=1}^{\infty}\exp(-\lambda_i^2\tau)\{b_i\exp(-\lambda_i^2\tau) + 2c_i\}}\,, \tag{20}$$

where $f(\eta) = \{1/4 - 3\eta^2/4 - \eta^4\ln\eta\}/\{(1+\eta)(1-\eta)^3\}$, and b_i and c_i are $b_i(\lambda_i, \eta) = \frac{k_i^2}{2}\left\{\left(Z_1'\left(\frac{\lambda_i}{1-\eta}\right)\right)^2 - \left(\eta Z_1'\left(\frac{\eta\lambda_i}{1-\eta}\right)\right)^2\right\}$, and $c_i(\lambda_i, \eta) = \frac{k_i}{\lambda_i}Z_0\left(\frac{\lambda_i}{1-\eta}\right)$.

For the limiting case of $\eta \to 1$, σ_0 is obtained from Eq. (6) as

$$\sigma_0 = \frac{8\sum_{i=0}^{\infty}\exp(-n^2\pi^2\tau)\{1 - \exp(-n^2\pi^2\tau)\}}{1/3 - \sum_{n=1}^{\infty}\exp(-n^2\pi^2\tau)/(n\pi)^2\{2 - \exp(-n^2\pi^2\tau)\}}\,. \tag{21}$$

For the case of $\tau \to \infty$, the above stability equations with $\sigma_0 = 0$ degenerate into the strong stability formulation. And, for the limiting case of $\tau \to 0$, based on the velocity profile of Eq. (7), it is found that $\sigma_0 = (1/E_0)(dE_0/d\tau) = 1/2\tau$ and therefore, the terms containing $\sigma_0/2$ should be changed as $1/4\tau$.

2-3. Solution Method

The stability equations (17)-(19) are solved by employing the shooting method[15]. In order to integrate these stability equations the proper values of $\partial^2u'/\partial r^2$, $\partial^3u'/\partial r^3$ and $\partial v'/\partial r$ at r = 1 are assumed for a given τ and a. Since the stability equations and their boundary conditions are all homogeneous, the value of $\partial^3u'/\partial r^3$ at r = 1 can be assigned arbitrarily and the value of the parameter Re is assumed. This procedure can be understood easily by taking into account the characteristics of eigenvalue problems. After all the values at r = 1 are provided, this eigenvalue problem can be proceeded numerically. Integration is performed from r = 1 to r = 0 with the fourth order Runge-Kutta-Gill method. If the guessed values of Re, $\partial^3u'/\partial r^3$ and $\partial v'/\partial r$ at r = 1 are correct, u′, $\partial u'/\partial r$ and v′ will vanish at r = 0. The minimum Re-value is found in the plot of Re vs. a.

3. Results and Discussion

The stability conditions obtained from the present relative and the

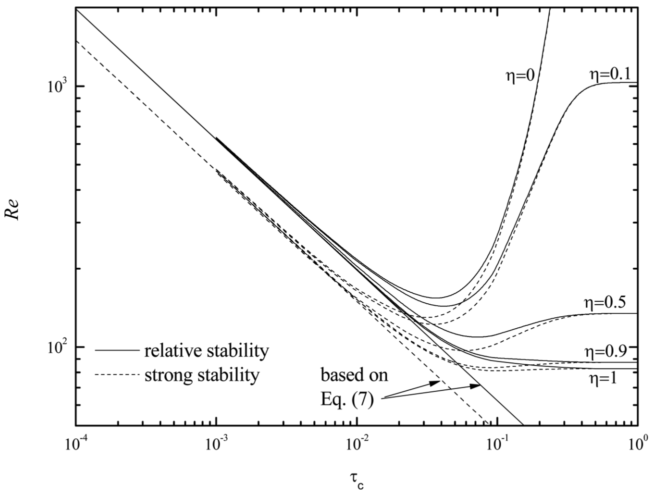

Fig. 3. Characteristic stability curves in the Re-τ diagram.

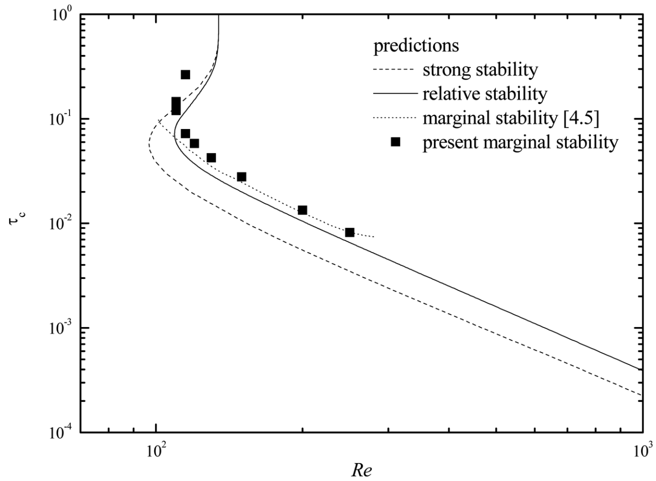

Fig. 4. Comparison among predictions for η = 0.5.

conventional strong stability model are illustrated in Fig. 3. It is known that the approximation (8) produces the same values as those from Eq. (5) as time decreases. As discussed below Eq. (19), the present relative stability model yields the strong stability limit as $\tau\to\infty$. For the limiting case of $\tau\to\infty$, the stability equations (15)-(17) are reduced to

$$-\frac{1}{2}a^2 \mathrm{Re}\,v' = \left\{\frac{d^2}{dr^2} - a^2\right\}u' = 0\,, \tag{22}$$

$$\frac{1}{2}\mathrm{Re}\,u' = \left\{\frac{d^2}{dr^2} - a^2\right\}v' = 0\,, \tag{23}$$

under the very narrow gap condition, *i.e.* $\eta\to 1$, where $\phi\to 1$ and $\sigma_0\to 0$ from Eqs. (5) and (19), respectively. The proper boundary conditions are

$$u' = \frac{\partial u'}{\partial r} = v' = 0 \;\text{ at r = 0 and 1.} \tag{24}$$

According to the calculation of Chandrasekhar's[16], the critical condition is $(\mathrm{Re}_S/2)^2 = 1{,}708$ and $a_c = 3.117$, *i.e.* $\mathrm{Re}_S = 82.66$ for $\eta\to 1$, here Re_S is the steady state critical Reynolds number. And, for the another limiting case of small τ, the critical time to mark the onset of a fastest growing instability decreases with increasing Re. Based on the base velocity field given in Eq. (7), they approach :

$$\tau_r = 389.27\mathrm{Re}^{-2} \;\text{ as } \tau\to 0, \tag{25a}$$

for the relative stability, and

$$\tau_s = 223.80\mathrm{Re}^{-2} \;\text{ as } \tau\to 0, \tag{25b}$$

for the strong stability, as are illustrated in Fig. 3.

For the case of a rather wide gap $\eta \le 0.5$, there exists the subcritical region where the global minimum of the critical Reynolds number, Re_G is lower than Re_S. In the range of $\mathrm{Re}_G \le \mathrm{Re} \le \mathrm{Re}_S$, the instability is transient and decays out as $\tau\to\tau_f$, here τ_f is the decay time from which the instability cannot be guaranteed. Therefore, for the region of $\mathrm{Re} > \mathrm{Re}_G$ the curves in Fig. 3 correspond to the decay time. Neitzel[4] and Chen and Neitzel[5] analyzed this problem by employing the marginal stability criterion. As shown in Fig. 4, their marginal stability results shift the strong stability curve to the more stable direction. However, they found the stability limits for the limited range. For the limiting case of $\eta\to 0$, the system is unconditionally stable and all instabilities are transient, *i.e.* all instabilities should be disappeared at a certain time. Even though τ_f has been predicted for the case of $\mathrm{Re}_G < \mathrm{Re}_S$, it has not been determined experimentally even for the asymtoptically unconditionally-stable case of $\eta\to 0$.

For the case of $\eta = 0.5$ the above results are compared with the predictions in Fig. 4. Neitzel's[4] marginal stability gives slightly more stable results than the present relative stability ones. Following the Neitzel's[4] idea, we have retried to find τ_m. First, for a given Re we found σ which satisfys Eq. (11) and constructed the relationship of σ vs. τ by employing a regression analysis. As shonwn in Fig. 5, the regression equation $\sigma_r(\tau)$ represents the calculated results quite well. Then, the marginal stability time τ_m is obtained by integrating the regression equation $\sigma_r(\tau)$ numerically. For the region of $\mathrm{Re} \ge 150$,

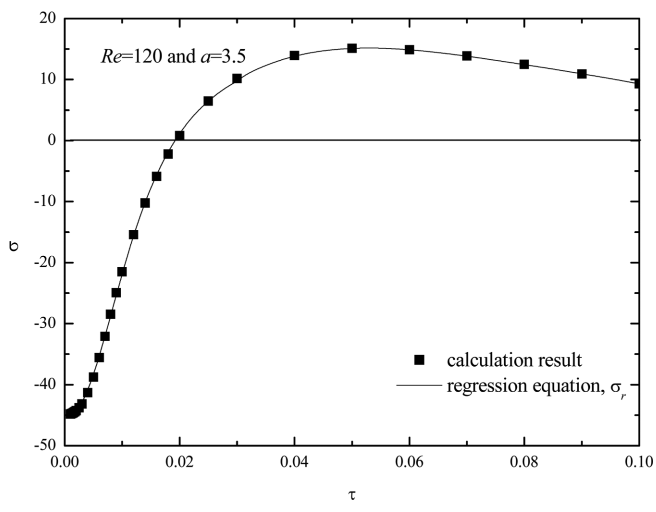

Fig. 5. Comparison of calculated growth rate with its regression function.

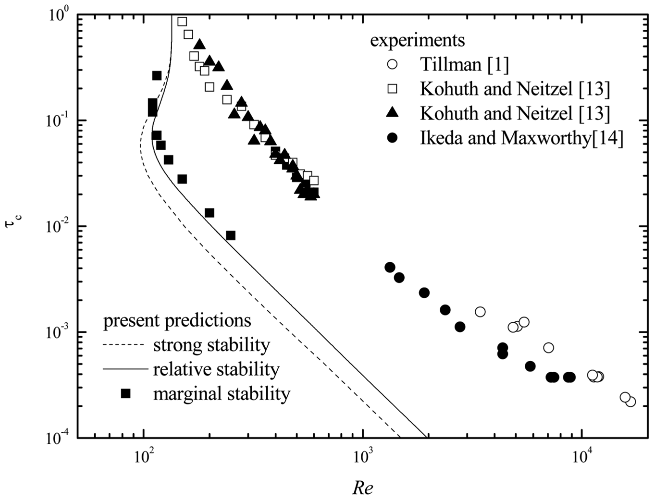

Fig. 6. Comparison of predictions $\eta = 0.5$ with experimental data: ○, $\eta = 0.625$ (Tilman[1]); □, $\eta = 0.5$ (visual observation, Kohuth and Neitzel[13]); ▲, $\eta = 0.5$ (photo diode array, Kohuth and Neitzel[13]); ●, $\eta = 0.464$ (Ikeda and Maxworthy[14]).

the present τ_m reconstructs Neitzel's [4] τ_m. However, for the region of Re < 110, we cannot find τ_m satisfying Eq. (10). The integration given in Eq. (10) based on the present $\sigma_r(\tau)$ is always negative for Re < 110, *i.e.,* $Re_{G,m} = 110$ for the case of $\eta = 0.5$ under the marginal stability concept. The present $Re_{G,m}$ is slightly higher than Chen and Neitzel's[5] global minumum Reynolds number, $Re_{G,m} = 102$. This minor difference may be come from the method to calculate the definite integral given in Eq. (10). To calculate that integral, Neitzel[4] used 3-points Gausse-Legendre quadrature which gives exact definite integal value only for a polynomial equation of the 2nd degree. The present $Re_{G,m}$ is nearly same as the global minumum Reynolds number based on the relative stability concept $Re_{G,r} = 109$.

The above results are compared with the available experimental data in Fig. 6. By using the suspended particle Tillman[1] visualized Taylor-Görtler vortex motion for the case of $\eta = 0.625$. Later, Kohuth and Neitzel[13] determined the onset time systematically by a photodiode array and visual observations for the experimental set-up of $\eta = 0.5$. Ikeda and Maxworthy[14] visualized the onset of vortex by adding aluminum flake into water placed between two coaxial cylinders of $\eta = 0.464$. As expected, none of the above stability criteria predict the onset time observed experimentally, as shown in Fig. 6. This may be caused by several factors. The time for disturbances to grow to finite amplitude before being observed seems to be a major one. Furthermore, the transient stability region was not observed in all the experiments, as mentioned above.

4. Conclusions

The onset of a fastest growing, axisymmetric instability in transient spin-down flow has been investigated theoretically. The strong stability results give the lower bounds on the stability limits, and the present relaxation of the relative instability shifts the stability limit to the more stable direction for the whole time, as expected. All the predicted critical times to mark the onset of vortices, which are shown in this study, are much smaller than available experimental data. It seems evident that disturbances require some growth period until they are detected experimentally.

Acknowledgement

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