A Simple Method to Make the Quadruple Tank System Near Linear

**Jietae Lee[†], Inhyun Kyoung, Jea Pil Heo, YoungSu Park, Yugyeong Lim, Dong Hyun Kim,
Yongjeh Lee* and Dae Ryook Yang***

Department of Chemical Engineering, Kyungpook National University, 80, Daehak-ro, Buk-gu, Daegu, 41566, Korea
*Department of Chemical and Biological Engineering, Korea University, 145, Anam-ro, Seongbuk-gu, Seoul, 02841, Korea
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Abstract – Quadruple tank liquid level systems are popular in testing multivariable control systems for multivariable processes with positive or negative zeros. The liquid level system is nonlinear and it will help to illustrate the robustness of control systems. However, due to nonlinearity, it can be cumbersome to obtain process parameters for testing linear control systems. Perturbation sizes are limited for valid linearized process models, requiring level sensors with high precision. A simple method where the outlet orifice is replaced to a long tube is proposed here. The effluent flow rate becomes proportional to the liquid level due to the friction loss of long tube and the liquid level system shows near linear dynamics. It is applied to the quadruple tank system for easier experiments.

Key words: Quadruple tank system, liquid level system, linear experimental system, Hagen-Poiseuille equation

1. Introduction

Laboratory experiments are important for engineering courses, especially, for process control courses [1]. Understanding dynamics (non-steady state responses) is required for process control and, for this, laboratory experiments are very useful [2]. Liquid level systems are one of the units widely used for process control experiments. They are simple, cheap and safe and no wastes are produced.

Liquid level systems are often too simple to illustrate various process dynamics. On the other hand, the quadruple tank system suggested by Johansson [3] breaks such negative views of liquid level systems without adding additional complicate apparatus. The quadruple tank system is a nonlinear multi-input and multi-output process that can be made to show non-minimum phase behavior, a nontrivial control problem to be resolved. Many research papers have been reported [4-12]. Several modifications have also been made [13-15]. Gatzke *et al.* [16] showed that the quadruple tank system is better than other experimental systems such as the inverted pendulum in the student reputations.

The liquid level system with an orifice outlet is nonlinear because the effluent flow rate is proportional to the square root of liquid level. This nonlinearity is useful to illustrate robustness of linear control systems. However, nonlinearity can limit sizes of the set point change and perturbation. Level sensors with high precision may be needed. To relieve these disadvantages, a simple method where the outlet orifice is replaced by a long tube is proposed. The liquid level system with a long tube outlet can be made to be near linear. Results are veri-

[†]To whom correspondence should be addressed.
E-mail: jtlee@knu.ac.kr

fied experimentally. It can be applied to the quadruple tank system for easier experiments. Linear experimental systems will reduce time in steps required to apply control researches such as the model identification step. Although the proposed system is near linear, robustness issues also exist because liquid flow rates can change as experimental environments such as ambient temperature and tube condition change.

2. Liquid Level System with Near Linear Dynamics

Consider a single tank system as shown in Fig. 1. For an orifice outlet (Fig. 1b), the dynamic model for liquid level is

$$A\frac{dh}{dt} = q_{in} - q_{out} = q_{in} - \eta a\sqrt{2gh} \qquad (1)$$

where A is the cross-sectional area of the tank, h is the tank level, q_{in} and q_{out} are input and output flow rates, g is the gravitational acceleration constant (980 cm/s^2), a is the cross-sectional area of the orifice and η is a correction factor. For q_{out} in Eq. (1), we use the relationship derived from the Bernoulli equation (or Torricelli's law) [17],

$$q_{out} = \eta a\sqrt{2gh} \qquad (2)$$

In almost all process control examples since Johansson [3], η is omitted ($\eta = 1$). However, without η, q_{out} can have errors not negligible for some orifice shapes. For example, for sharp edge orifices such as the drilled holes and tube fittings, $\eta = 0.62$ should be used [17] and, in this case, q_{out} with $\eta = 1$ has 61% error.

The linearized model is

$$A\frac{dh}{dt} = bu - \frac{\eta a\sqrt{g}}{\sqrt{2}\sqrt{h_{ss}}}y, (y = h - h_{ss}; u = (q_{in} - q_{in,ss})/b) \qquad (3)$$
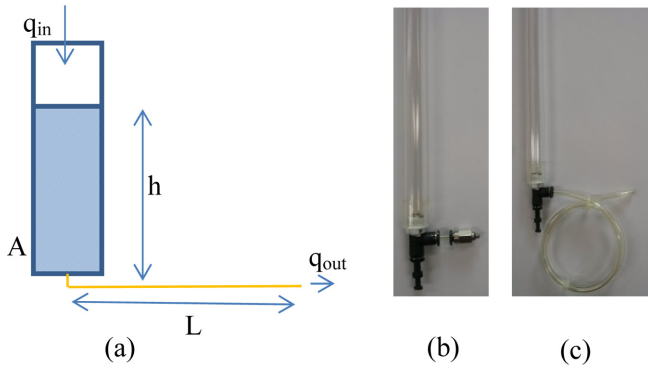
Fig. 1. A liquid level system ((a) schematic diagram, (b) horizontal orifice, and (c) tube coiled two times).

Here the subscript ss denotes the steady state value. The process gain k and time constant τ in the linearized first order model of Eq. (3) are

$$k = \frac{b\sqrt{2}\sqrt{h_{ss}}}{\eta a \sqrt{g}}, \tau = \frac{A\sqrt{2}\sqrt{h_{ss}}}{\eta a \sqrt{g}} \qquad (4)$$

The process gain and time constant vary much as the operating point of h_{ss} varies.

For a near linear dynamics, a long tube as in Fig. 1c is considered. A laminar flow [17] is assumed such that

$$N_{Re} = \frac{\rho D V}{\mu} < 2400 \qquad (5)$$

where N_{Re} is the Reynold number (dimensionless), ρ is the liquid density (1.0 g/cm^3 for water at 20 ^{o}C), D is the inner diameter (ID) of tube, V is the mean velocity of liquid and μ is the viscosity of liquid (0.01 g/(cm·s) for water at 20 ^{o}C). For a horizontal straight tube with the length L, the Bernoulli equation becomes

$$gh = \frac{1}{2}V^2\left(1 + 4\frac{16}{N_{Re}}\frac{L}{D}\right) = \frac{1}{2}V^2 + \frac{32\mu L}{\rho D^2}V \qquad (6)$$

The friction loss in the last term of Eq. (6) is due to the Hagen-Poiseuille equation [17]. For a low liquid velocity of V, ignoring the quadratic term in Eq. (6), we have

$$V \approx \frac{\rho g D^2}{32\mu L}h \qquad (7)$$

and

$$q_{out} = aV \approx \frac{\pi \rho g D^4}{128\mu L} \qquad (8)$$

The dynamic model for the tank level becomes, with introducing a correction factor ξ,

$$A\frac{dh}{dt} = q_{in} - \xi\frac{\pi \rho g D^4}{128\mu L}h \qquad (9)$$

This model is linear and its process gain k and time constant τ are

$$k = \frac{128b\mu L}{\xi\pi\rho g D^4}, \tau = \frac{128A\mu L}{\xi\pi\rho g D^4} \qquad (10)$$
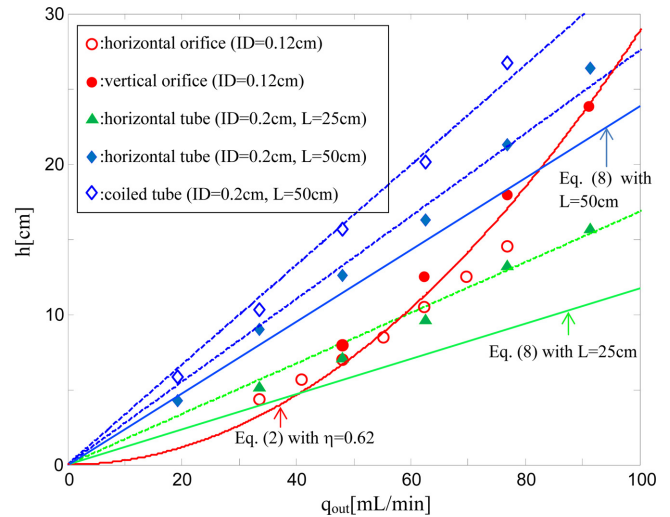
Fig. 2. Liquid levels versus effluent flow rates for various outlet tubes (experimental results).

They are independent of the operating point of h_{ss}. Experiments can be carried out without worrying that the time constant and process gain become too small or large.

Figure 2 shows experimental results to illustrate the Bernoulli equation (2) and the Hagen-Poiseuille equation (8). Water with $\rho = 1$ g/cm^3 and $\mu = 1.1$ cP $= 0.011$ g/(cm·s) is used for the working liquid. The Bernoulli equation (2) describes the effluent flow rate well when $\eta = 0.62$ is used. For long tubes, that relationships between the tank level and effluent flow rate are near linear. However, the Hagen-Poiseuille equation (8) has some errors and their causes are beyond our considerations here. As in the Bernoulli equation (2), the correction factor ξ should be used and ξ (or ξa) should be identified experimentally.

Figure 3 shows step responses for various steady state levels whose perturbation sizes are all the same. The tank ID is 7.5 cm. The orifice ID is 0.2 cm and the long tube is 2 m long with ID of 0.4 cm. For the
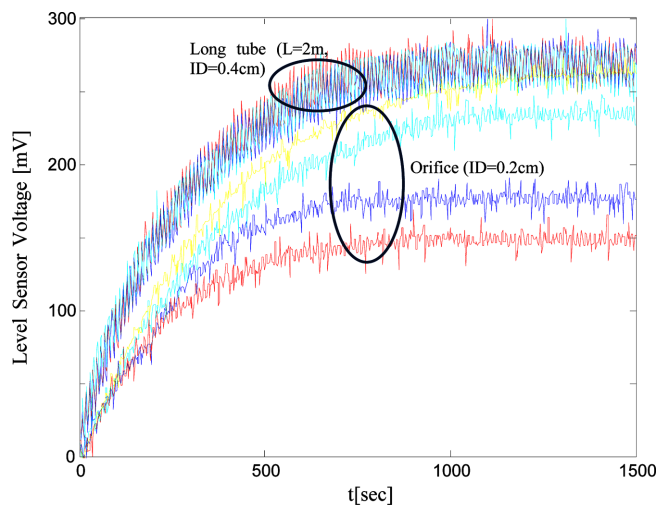

Fig. 3. Experimental step responses for various steady state levels where perturbation sizes of the input flow are all the same (For the long tube, step responses from three different steady state levels are overlapped).
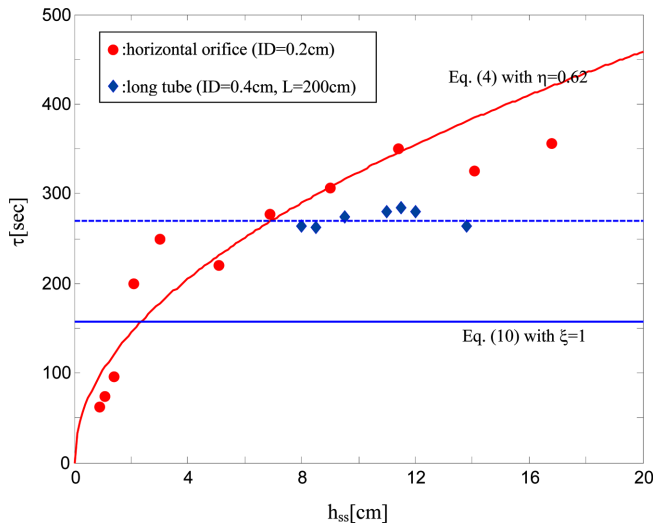
Fig. 4. Time constants (Tank ID=7.5 cm, experimental results).

orifice outlet, the sizes and speeds of level changes are varying according to the initial steady state level. On the other hand, they are not varying for the long tube outlet (Step responses from three different steady state levels are nearly overlapped in Fig. 3).

Figure 4 shows time constants for tanks with the orifice outlet and long tube. The time constant is obtained by finding the time for the liquid level to reach 63.2% of the new steady state value graphically for a step change of influent flow rate. For the orifice outlet, the time constant is growing by the rate of $\sqrt{h_{ss}}$ as the operating point h_{ss} grows. On the other hand, for the long tube outlet, the time constant is not varying and is independent of the operating point h_{ss}.

3. Quadruple Tank Systems

3-1. Quadruple Tank System in the Non-interacting Connection

Consider a quadruple tank system in Fig. 5a suggested by Johansson [3]. Four tanks have the same cross-sectional area A with the same long tube outlet with the inner diameter D and length L. Then

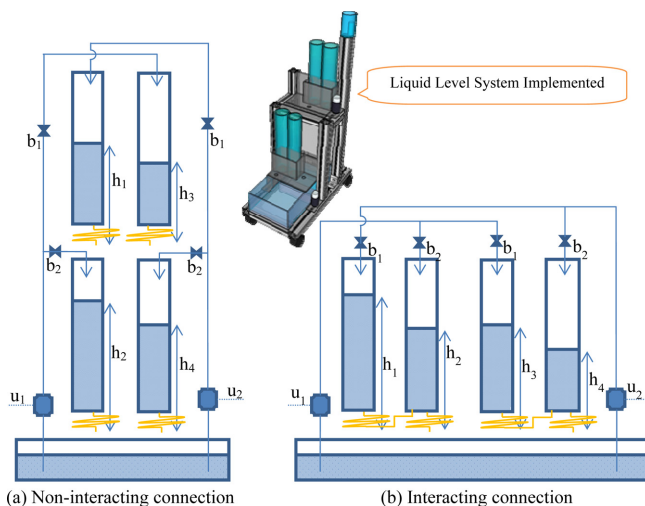(a) Non-interacting connection   (b) Interacting connection

Fig. 5. Quadruple tank systems with long tube outlets.

the dynamic model becomes

$$A\frac{dh_1}{dt} = b_1 u_2 - \gamma h_1$$

$$A\frac{dh_2}{dt} = b_2 u_1 + \gamma h_1 - \gamma h_2$$

$$A\frac{dh_3}{dt} = b_2 u_1 - \gamma h_3$$

$$A\frac{dh_4}{dt} = b_2 u_2 + \gamma h_3 - \gamma h_4 \, , \ \ \gamma = \xi\frac{\pi \rho g D^4}{128\mu L}$$

$$y_1 = h_2$$

$$y_2 = h_4 \tag{11}$$

Here b_1 and b_2 are valve coefficients for the input flow control signals of u_1 and u_2. Process parameters to be identified experimentally are few and, in addition, they are independent of four liquid levels.

Dynamic model of Eq. (11) is linear and its transfer function between the input variables $u=(u_1, u_2)^T$ and the output variables $y= (y_1, y_2)^T$ is

$$Y(s) = G_a(s)U(s)$$

$$G_a(s) = \begin{pmatrix} \dfrac{b_2}{As+\gamma} & \dfrac{b_1\gamma}{(As+\gamma)^2} \\ \dfrac{b_1\gamma}{(As+\gamma)^2} & \dfrac{b_2}{As+\gamma} \end{pmatrix} \tag{12}$$

The zeros of the transfer function matrix $G_a(s)$ are the zeros of the numerator polynomial of

$$\det(G_a(s)) = \frac{b_2^2}{(As+\gamma)^2} - \frac{b_1^2\gamma^2}{(As+\gamma)^4} = \frac{b_2^2(As+\gamma)^2 - b_1^2\gamma^2}{(As+\gamma)^4} \tag{13}$$

Hence, when $b_1 > b_2 > 0$, the numerator polynomial of Eq. (13) has a positive real root and the liquid level system of Eq. (11) will show the non-minimum phase behavior [3].

The process gain matrix K is

$$K = G_a(0) = \frac{1}{\gamma}\begin{pmatrix} b_2 & b_1 \\ b_1 & b_2 \end{pmatrix} \tag{14}$$

The relative gain array becomes

$$\Lambda = \frac{1}{b_2^2 - b_1^2}\begin{pmatrix} b_2^2 & -b_1^2 \\ -b_1^2 & b_2^2 \end{pmatrix} \tag{15}$$

Considering the transfer function of Eq. (12), y_1 is dynamically preferred to be controlled by u_1 and, similarly, y_2 by u_2. However, when $b_1 > b_2 > 0$, the pairing of $(u_1 - y_1)$ and $(u_2 - y_2)$ for the multi-loop control cannot be used due to the negative relative gain [18].

The relative gain array of Eq. (15) is dependent only on b_i's. Hence a given relative gain array is realized easily. However, when the usual orifices are used instead of the long tubes, it is dependent on the steady state liquid levels in addition to b_i's. So it will be very difficult to obtain the operating conditions where the four tank system

has a given relative gain array.

3-2. Quadruple Tank System in the Interacting Connection

Consider the quadruple tank system in Fig. 5b used in [15]. This four tank system is somewhat easier to construct than the original one of Fig. 5a because all tanks are located at the same elevation. For the same tanks and outlet tubes as above, the dynamic model becomes

$$A\frac{dh_1}{dt} = b_1 u_2 - \gamma(h_1 - h_2)$$

$$A\frac{dh_2}{dt} = b_2 u_1 + \gamma(h_1 - h_2) - \gamma h_2$$

$$A\frac{dh_3}{dt} = b_1 u_1 - \gamma(h_3 - h_4)$$

$$A\frac{dh_4}{dt} = b_2 u_2 + \gamma(h_3 - h_4) - \gamma h_4 \, , \quad \gamma = \xi\frac{\pi\rho g D^4}{128\mu L}$$

$$y_1 = h_2$$

$$y_2 = h_4 \tag{16}$$

The dynamic model of Eq. (16) is linear and its transfer function between the input variables $u=(u_1, u_2)^T$ and the output variables $y=(y_1, y_2)^T$ is

$$Y(s) = G_b(s)U(s)$$

$$G_b(s) = \frac{1}{A^2 s^2 + 3\gamma As + \gamma^2}\begin{pmatrix} b_2(As+\gamma) & b_1\gamma \\ b_1\gamma & b_2(As+\gamma) \end{pmatrix} \tag{17}$$

The zeros of the transfer function matrix $G_b(s)$ are the zeros of the numerator polynomial of

$$\det(G_b(s)) = \frac{b_2^2(As+\gamma)^2 - b_1^2\gamma^2}{(A^2 s^2 + 3\gamma As + \gamma^2)^2} \tag{18}$$

Hence, when $b_1 > b_2 > 0$, the numerator polynomial of Eq. (18) has a positive real root and the liquid level system of Eq. (16) will show the non-minimum phase behavior. The process gain matrix $K=G_b(0)$ of the system (17) is the same as Eq. (14). When $b_1 > b_2 > 0$, the pairing of (u_1-y_1) and (u_2-y_2) for the multiloop control cannot be used.

4. Conclusions

Liquid level systems are widely used in testing process control researches and teaching in the process control laboratory. The liquid level system where the outlet orifice is replaced to a long tube is shown to have near linear dynamics. When the effluent flow is laminar, a long tube produces friction and its flow rate is proportional to the liquid level (Hagen-Poiseuille equation [17]). Applying this technique, the quadruple tank system can be made to be near linear with

keeping properties about zero locations and relative gain arrays. The proposed modification can help the liquid level system be used in the process control laboratories with ease and more confidence.

Acknowledgments

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