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Approximation of the Optimal Minimum-phase Output for Control of Nonlinear Nonminimum-phase Processes

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Abstract−Nonminimum-phase parts are better removed in the feedback loop like the time delay term. For this, Wright and Kravaris [1992] proposed the concept of optimal minimum-phase output to control nonlinear nonminimum-phase processes. However, their optimal minimum-phase output has no analytic solutions for processes with more than three state variables. Here, methods for analytic minimum-phase outputs approximating the optimal ones are proposed, having no limitations in the number of state variables. The proposed methods provide analytic solutions for processes with three state variables and simple numerical solutions for those with more state variables.

Key words: Nonlinear Nonminimum-phase Process, Input/Output Linearization, ISE Optimal Minimum-phase Output

INTRODUCTION

For nonlinear nonminimum-phase processes, controller design methods such as the input/output linearization method [Kravaris and Chung, 1987], which are based on the inversion of process dynamics, cannot be applied directly due to their unstable inverse. The nonminimum-phase parts which yield unstable inverse should be removed in the inversion mechanism. For linear processes, the nonminimum-phase parts can be easily isolated by decomposing the process transfer functions. The internal model control of Garcia and Morari [1982] and the generalized Smith predictor of Ramanathan et al. [1989] incorporate decomposition of transfer functions in their controller design frameworks. For nonlinear nonminimum-phase processes, such decomposition is not apparent except for the second order processes [Kravaris and Daoutidis, 1990]. Instead, Wright and Kravaris [1992] have proposed a method using auxiliary outputs which are statically equivalent to the real outputs and minimum-phase. With the auxiliary outputs, minimum-phase predictors and consequent nonlinear control systems for the nonlinear nonminimum-phase processes can be designed. One of key steps in the method is how to obtain appropriate statically equivalent minimum-phase outputs. For this, Wright and Kravaris [1992] also have proposed statically equivalent outputs which are optimal with respect to a certain integral of square error (ISE) criterion. Their control systems provide excellent control performances. However, obtaining the ISE optimal minimum-phase output is somewhat restrictive because it is based on the nonlinear processes in natural coordinates [Hunt et al., 1983]. Furthermore, because it requires solving high-order Euler-Lagrange equations for nonlinear dynamical optimization, it is very hard to compute except for the processes with zero dynamics of order 2 for which analytic solutions exist.

Here we propose a statically equivalent minimum-phase output which approximates the ISE optimal minimum-phase one of Wright and Kravaris [1992] and can be used in the framework of their control system for nonlinear nonminimum-phase processes. The extended linearization method [Baumann and Rugh, 1986; Lin, 1994] is utilized. As shown later, it is the first order Taylor series expansion of the global ISE optimal minimum-phase one which is valid for neighborhoods along equilibrium points. If set point changes are not very fast and control actions are mild, states of the process will not excurse far from their equilibrium values and the proposed minimum-phase output approximating the global ISE optimal one will be effective. It has the following properties:

- Analytic solutions are available for processes with zero dynamics of orders up to 3.
- For processes with higher order zero dynamics, solutions can be easily calculated numerically and interpolated to analytic forms.
- When linearized at equilibrium points, both the proposed output and the ISE optimal minimum-phase output are the same. Minimum-phase predictor with the proposed output can be interpreted as a gain-scheduled linear ISE optimal predictor compensating variation of the equilibrium point because it is the linear approximation of the global ISE optimal output of Wright and Kravaris [1992].

1. Nonlinear Nonminimum-phase Processes

Consider a nonlinear process

$$x(t)=f(x(t))+g(x(t))u(t)$$

$$y(t)=h(x(t)) \tag{1}$$

where x, u and y are the n state variable vector, the scalar input variable and the scalar output variable, respectively. It is assumed that the process (1) has an isolated equilibrium point (u_s, x_s). The process (1) is locally minimum-phase in a neighborhood of the equilibrium point (u_s, x_s), if all roots of the following polynomial in Laplace variable s are in the open left half plane:

$$n(s)=c\ \mathrm{adj}(sI-A)\ b \tag{2}$$

where

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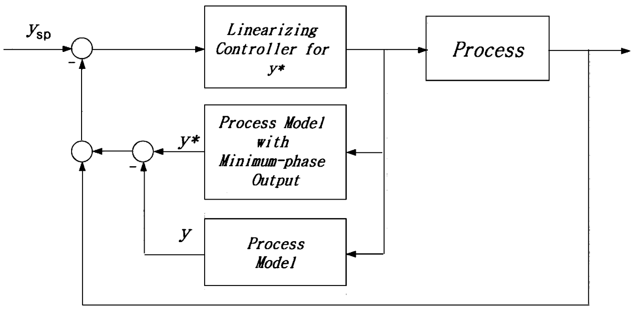

Fig. 1. Linearizing control system for nonlinear nonminimum-phase processes.

$$A=\left[\frac{\partial f(x)}{\partial x}+\frac{\partial g(x)}{\partial x}u_s\right]_{x_s},\ b=g(x_s),\ c=\left[\frac{\partial h(x)}{\partial x}\right]_{x_s}$$

Processes for which some roots of (2) are in the open right half plane over desired equilibrium points are considered. The input/output linearization method cannot be applied directly to such processes. Wright and Kravaris [1992] proposed a nonlinear control system as in Fig. 1 which is an extension of the linear generalized Smith predictor. In Fig. 1, the auxiliary output

$$y^*=h^*(x) \tag{3}$$

is such that it is statically equivalent to the output y and minimum-phase. The choice of the auxiliary output y* is rather arbitrary but very much affects control performance. With transforming the process in natural coordinates, Wright and Kravaris [1992] proposed the auxiliary output which is optimal with respect to the integral of square error (ISE) between y and y* when controlled perfectly. The ISE optimal minimum-phase output provides excellent control performance. However, because it requires solving the Euler-Lagrange equation, it is very hard to obtain except for the second order case. Its approximation is investigated here.

2. Approximation of the ISE Optimal Mimimum-phase Output

The ISE optimal minimum-phase output of Wright and Kravaris [1992] can be obtained for a nonlinear process in natural coordinates [Hunt et al., 1983]. Consider a nonlinear system of relative order r in natural coordinates:

$$\begin{aligned}
&\zeta_1=\zeta_2\\
&\zeta_2=\zeta_3\\
&\vdots\\
&\zeta_{n-1}=\zeta_n\\
&\zeta_n=\varphi(\zeta_1, ..., \zeta_n)+\psi(\zeta_1, ..., \zeta_n)u\\
&y=h(\zeta_1, ..., \zeta_{n-r+1})
\end{aligned} \tag{4}$$

The local zero polynomial around an equilibrium point $\zeta_s=(\zeta_{1s}, 0, ..., 0)^T$ is

$$n(s)=\frac{\partial h(\zeta_s)}{\partial\zeta_1}+\frac{\partial h(\zeta_s)}{\partial\zeta_2}s+\cdots+\frac{\partial h(\zeta_s)}{\partial\zeta_{n-r+1}}s^{n-r} \tag{5}$$

If some roots of the zero polynomial (5) are in the open right half plane, control systems like in Fig. 1 which does not inverse the zero dynamics should be used. For the control system in Fig. 1, the ISE optimal minimum-phase output y* is proposed by Wright and Kravaris [1992]. It is stated as an (n−r+1)-th dimensional nonlinear dynamical optimization problem and it results in solving an (n−r+1)-th order Euler-Lagrange equation which is usually very hard to solve except for the second order case. Minimum-phase output which approximates the ISE optimal one is investigated. Since the steady-state values of states, ζ_i, i=2, 3, ..., n−r+1, are zero, the first approximation which is valid along equilibrium points is

$$y^*=h_1^*(\zeta_1)+h_2^*(\zeta_1)\zeta_2+\cdots+h_{n-r+1}^*(\zeta_1)\zeta_{n-r+1} \tag{6}$$

It is just the first order Taylor series expansion about for ζ_i, i=2, 3, ..., n−r+1 around their steady-state values of zero. From Eq. (4), we can see that the state variables ζ_i, i=2, 3, ..., n−r+1 will not be far from their steady-state values of zero if the control input is mild. Hence effects of higher order terms ignored in Eq. (6) will not be much.

The zero polynomial for the auxiliary output (6) around an equilibrium point $\zeta_s=(\zeta_{1s}, 0, ..., 0)^T$ is

$$n^*(s)=\frac{dh_1^*(\zeta_{1s})}{d\zeta_1}+h_2^*(\zeta_{1s})s+\cdots h_{n-r+1}^*(\zeta_{1s})s^{n-r} \tag{7}$$

For the auxiliary output (6) to be approximation of the ISE optimal minimum-phase output, the zero polynomial (7) should be ISE optimal locally around the equilibrium point $\zeta_s=(\zeta_{1s}, 0, ..., 0)^T$ and statically equivalent. That is, it should satisfy [Wright and Kravaris, 1992]

$$n^*(s)\ n^*(-s)=n(s)\ n(-s) \tag{8}$$

and

$$h_1^*(\zeta_{1s})=h(\zeta_s)$$

From Eqs. (5), (7) and (8), we have [Riddle and Anderson, 1966]

$$\begin{aligned}
&\left[\frac{dh_1^*(\zeta_{1s})}{d\zeta_1}\right]^2=\left[\frac{\partial h(\zeta_s)}{\partial\zeta_1}\right]^2\\
&[\,h_2^*(\zeta_{1s})\,]^2-2\frac{dh_1^*(\zeta_{1s})}{d\zeta_1}h_3^*(\zeta_{1s})=\left[\frac{\partial h(\zeta_s)}{\partial\zeta_1}\right]^2-2\frac{\partial h(\zeta_s)}{\partial\zeta_1}\frac{\partial h(\zeta_s)}{\partial\zeta_3}\\
&\vdots\\
&[\,h_{n-r+1}^*(\zeta_{1s})\,]^2=\left[\frac{\partial h(\zeta_s)}{\partial\zeta_{n-r+1}}\right]^2
\end{aligned} \tag{9}$$

Argument ζ_{1s} in both sides of Eq. (9) is a dummy and can take any value in the desired whole range. Hence we can drop the subscript s in ζ_{1s}. Closed-form solutions are available for orders up to 3. For example, in case of n−r+1=2, the solution of (9) is

$$y^*=h(\zeta_1, 0)-\frac{\partial h(\zeta_1, 0)}{\partial\zeta_2}\zeta_2 \tag{10}$$

It can be easily checked that it is the first order Taylor series expansion about ζ_2=0 of the ISE optimal minimum-phase output of Wright and Kravaris [1992]

$$y_{WK}^*=h(\zeta_1, \zeta_2)-2\frac{\partial h(\zeta_1, \zeta_2)}{\partial\zeta_2}\zeta_2 \tag{11}$$

For n−r+1=3 and $\frac{\partial h(\zeta_1, 0, 0)}{\partial\zeta_1}>0$ (without loss of generality),

$$y^*=h(\zeta_1,0,0)+\sqrt{\begin{matrix}\left[\dfrac{\partial h(\zeta_1,0,0)}{\partial\zeta_2}\right]^2-2\dfrac{\partial h(\zeta_1,0,0)}{\partial\zeta_1}\dfrac{\partial h(\zeta_1,0,0)}{\partial\zeta_3}\\+2\dfrac{\partial h(\zeta_1,0,0)}{\partial\zeta_1}\left|\dfrac{\partial h(\zeta_1,0,0)}{\partial\zeta_3}\right|\end{matrix}}\;\zeta_2+\left|\dfrac{\partial h(\zeta_1,0,0)}{\partial\zeta_3}\right|\zeta_3 \tag{12}$$

There is no such closed-form solution for the ISE optimal minimum-phase output. When $n-r+1$ is greater than 3, solutions of (9) can be found numerically for the whole range of the state ζ_1 and can be interpolated to the analytical minimum-phase output.

3. Extension to General Nonlinear Nonminimum-phase Processes

The auxiliary output of the form (6) can also be applied to general nonlinear nonminimum-phase processes which cannot be transformed to the natural coordinates whenever the equilibrium point is of the form $\zeta_s=(\zeta_{1s}, 0, ..., 0)^T$. Since a restrictive class of nonlinear processes can be transformed to the process in natural coordinates, it is important for wide applications. For a nonlinear system (1), the equation $f(x)+g(x)u=0$ usually has a solution about $\{u, x_2, x_3, ..., x_n\}$ as a function of x_1. Let the solution for x_i as $\phi_i(x_1)$ and define new state variables as

$$\zeta_1=x_1$$

$$\zeta_i=x_i-\phi_i(x_i),\ i=2, 3, ..., n \tag{13}$$

Then the state Eq. (1) becomes

$$\zeta=\varphi(\zeta)+\psi(\zeta)u$$

$$y=h(\zeta) \tag{14}$$

with an equilibrium point of the form $\zeta_s=(\zeta_{1s}, 0, ..., 0)^T$.

The zero polynomial is

$$n(s)=c\ \mathrm{adj}(sI-A)b \tag{15}$$

where

$$A=\left[\frac{\partial\varphi}{\partial\zeta}+\frac{\partial\psi}{\partial\zeta}u_s\right]_{\zeta_s},\ b=\psi(\zeta_s),\ c=\left[\frac{\partial h}{\partial\zeta}\right]_{\zeta_s}$$

The zero polynomial for the process with output y* of the form (6) is

$$n^*(s)=c^*\ \mathrm{adj}(sI-A)b \tag{16}$$

where

$$c^*=\left[\frac{dh_1^*(\zeta_{1s})}{d\zeta_1}\ h_2^*(\zeta_{1s}) \ldots h_n^*(\zeta_{1s})\right]$$

For the output y* to be ISE optimal along equilibrium points, the above two zero polynomials should satisfy the Eq. (8). It is known as a spectral factorization problem to find a stable polynomial n*(s) from n(s) satisfying Eq. (8). There are closed-form solutions for orders up to 3. Numerical iterative methods for higher order of equations are given in Riddle and Anderson [1966]. From n(s) as a function of ζ_{1s} we obtain n*(s) by solving Eq. (8). With n*(s), c* and consequently $h_i^*(\zeta_1)$ can be easily found from (16) since they appear linearly.

EXAMPLES

1. Example 1

For illustration purposes, we consider a second order nonminimum-phase process as [Wright and Kravaris, 1992];

$$x_1=x_2$$

$$x_2=f(x_1, x_2)+g(x_1, x_2)u$$

$$y=h(x)=x_1-3x_2-x_1^2x_2-x_2^3$$

The process is already in the form of natural coordinates. The equilibrium point is $x_s=(x_{1s}, 0)$. The zero polynomial is

$$n(s)=\frac{\partial h(x_s)}{\partial x_1}+\frac{\partial h(x_s)}{\partial x_2}s=1-(3+x_{1s}^2)s$$

Hence it is nonminimum-phase along the equilibrium points. From Eq. (8), we have

$$n^*(s)=1+(3+x_{1s}^2)s$$

Therefore, $h_1^*(x_{1s})=x_{1s}$ and $h_2^*(x_{1s})=3+x_{1s}^2$. The minimum-phase output which results in the above zero polynomial at each equilibrium point is

$$y^*=x_1+3x_2+x_1^2x_2$$

We can see that it is the first order Taylor series expansion about x_2 of the global ISE optimal one of Wright and Kravaris [1992];

$$y_{WK}^*=x_1+3x_2+x_1^2x_2+5x_2^3$$

2. Example 2

Consider the system of a stirred tank reactor where the isothermal series/parallel Van de Vusse reaction is taking place. It is described in the state space form as:

$$x_1=-k_1x_1-k_3x_1^2+(c_{ao}-x_1)u$$

$$x_2=k_1x_1-k_2x_2-x_2u$$

$$y=x_2$$

where $k_1=50$, $k_2=100$, $k_3=10$, $c_{ao}=10$, $x_{1s}=3$ and $x_{2s}=1.117$ (see Wright and Kravaris [1992] for meanings and dimensions in detail). The transformation [Wright and Kravaris, 1992]

$$\zeta_1=\frac{c_{ao}-x_1}{x_2}$$

$$\zeta_2=\frac{k_1x_1+k_3x_1^2}{x_2}-\frac{(c_{ao}-x_1)(k_1x_1-k_2x_2)}{x_2^2}$$

leads the process to the natural coordinates form and the output becomes

$$y=\frac{-B+\sqrt{B^2-4AC}}{2A}$$

where

$$A=k_3\zeta_1^2=\frac{k_3c_{ao}^2-2k_3c_{ao}x_1+k_3x_1^2}{x_2^2}$$

$$B=-\zeta_2-\zeta_1(k_1-k_2+2k_3c_{ao}-k_1\zeta_1)$$
$$=\frac{k_1c_{ao}^2-k_1c_{ao}x_1-k_1c_{ao}x_2-2k_3c_{ao}^2x_2+2k_3c_{ao}x_1x_2-k_3x_1^2x_2}{x_2^2}$$

$$C = k_1c_{ao} + k_3c_{ao}^2 - k_1c_{ao}\zeta_1 = k_1c_{ao} + k_3c_{ao}^2 - k_1c_{ao}\frac{c_{ao}-x_1}{x_2}$$

Hence, from Eq. (10),

$$\begin{aligned} y^* &= \left[\frac{-B+\sqrt{B^2-4AC}}{2A}\right]_{\zeta_2=0} - \frac{\partial}{\partial\zeta_2}\left[\frac{-B+\sqrt{B^2-4AC}}{2A}\right]_{\zeta_2=0}\zeta_2 \\ &= \left[-B-2D+\frac{B^2+3BD+2D^2-4AC}{\sqrt{(B+D)^2-4AC}}\right]\Big/2A \end{aligned} \tag{17}$$

where

$$D = \zeta_2 = \frac{k_1x_1+k_3x_1^2}{x_2} - \frac{(c_{ao}-x_1)(k_1x_1-k_2x_2)}{x_2^2}$$

Since at the steady-state

$$u_s = \frac{k_1x_{1s}+k_3x_{1s}^2}{c_{ao}-x_{1s}}$$

$$x_{2s} = \frac{k_1x_{1s}}{k_2+u_s} = \frac{k_1x_{1s}}{k_2+(k_1x_{1s}+k_3x_{1s}^2)/(c_{ao}-x_{1s})}$$

we can find another transformation which results in the state Eq. (14) as

$$\zeta_1 = x_1$$

$$\zeta_2 = x_2 - \phi_2(x_1)$$

where

$$\phi_2(x_1) = \frac{k_1x_1(c_{ao}-x_1)}{k_2(c_{ao}-x_1)+k_1x_1+k_3x_1^2}$$

The output becomes

$$y = \phi_2(\zeta_1) + \zeta_2$$

Thus, we can choose, as an appropriate auxiliary minimum-phase output,

$$y^* = \phi_2(\zeta_1) + h_2^*(\zeta_1)\zeta_2$$

The zero polynomials for the output and the auxiliary minimum-phase output are

$$n(s) = \phi_2(\zeta_{1s})(k_2+u_s)(c_{ao}-\zeta_{1s}) - \phi_2(\zeta_{1s})s$$

$$\begin{aligned} n^*(s) = \phi_2(\zeta_{1s})(k_2+u_s)(c_{ao}-\zeta_{1s}) + \{\phi_2(\zeta_{1s})(c_{ao}-\zeta_{1s}) \\ - h_2^*(\zeta_{1s})[\phi_2(\zeta_{1s})(c_{ao}-\zeta_{1s}) + \phi_2(\zeta_{1s})]\}s \end{aligned}$$

Hence

$$h_2^*(\zeta_1) = \frac{\phi'_2(\zeta_1)(c_{ao}-\zeta_1) - \phi_2(\zeta_1)}{\phi'_2(\zeta_1)(c_{ao}-\zeta_1) + \phi_2(\zeta_1)}$$

That is,

$$y^* = x_2 - 2\frac{x_2-\phi_2(x_1)}{\dfrac{\phi'_2(x_1)}{\phi_2(x_1)}(c_{ao}-x_1)+1} \tag{18}$$

becomes a local ISE optimal minimum-phase output.

Control performances with the ISE optimal minimum-phase output of Wright and Kravaris [1992] and the above two auxiliary outputs (17) and (18) were compared. The same linearizing control system as in Wright and Kravaris [1992] with β=0.01, K_c=5 and τ_I=0.01 was used (Fig. 1). Integration step size was 0.002 and 500 steps were simulated. Partial derivatives for linearizing controller block in Fig. 1 were calculated numerically via the central difference method with perturbation of 0.0001 to avoid mistakes in the program coding. The integral of square errors between y_{sp} and y for load change and set-point change are shown in Table 1. Degradation due to our approximation is not so serious and would be compensated by adjusting parameters of the external PI controller. Control responses for step set-point changes are shown in Fig. 2. All of them are almost not distinguishable.

Table 1. Integral of square errors between y_{sp} and y for load and set-point changes

Change	Wright and Kravaris	Output (17)	Output (18)
Load (c_{a0} : 10 to 9)	6.244E-5	6.364E-5	6.565E-5
Set-point (1.117 to 1.05)	5.478E-5	5.445E-5	5.392E-5

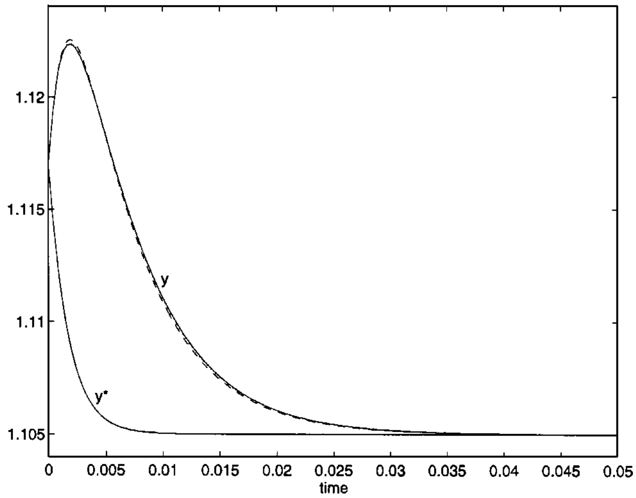

Fig. 2. Control responses with three minimum-phase outputs in case of a step set-point change of the Van de Vusse reaction example: solid line-Wright and Kravaris ISE optimal output, dotted line-output (17), and dashed line-output (18).

CONCLUSION

Methods to obtain minimum-phase outputs approximating the ISE optimal minimum-phase outputs of Wright and Kravaris [1992] are proposed. While the Wright and Kravaris method has no analytic solutions for processes with more than three state variables, the proposed method has no limitations in the number of state variables.

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