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Oscillatory Convection in a Horizontal Porous Layer Saturated with a Viscoelastic Fluid

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Abstract−When a horizontal porous layer saturated with a viscoelastic liquid is heated from below, the onset conditions of thermal convection are found to be functions of Darcy-Rayleigh number, wave number, and viscoelastic properties. In this study, the linear stability was studied analytically in order to investigate the viscoelastic effects of saturated liquids on the onset conditions in connection with oscillatory instabilities at the threshold of stationary convection. It is suggested that the resulting oscillatory instabilities occur at lower values of Darcy-Rayleigh number than the critical value for the stationary convection. From the occurrence of oscillatory instabilities of viscoelastic liquid, it is expected that the periodic motion should be replaced by stationary modes in a horizontal porous layer.

Key words: Porous Layer, Viscoelastic Liquid, Oscillatory Instabilities, Stationary Convection, Darcy-Rayleigh Number

INTRODUCTION

Buoyancy-driven convection in porous media has recently received much attention in connection with engineering applications. It is well known that the onset of convective motion in a fluid-saturated porous medium is governed by the Darcy-Rayleigh number which represents the buoyancy effects. Therefore, proper understanding of subsequent convective behavior is important for controlling many processes such as geothermal reservoirs, catalytic packed bed, filtration, enhanced oil recovery, polymer filament package and composite impregnations.

Horton and Rogers [1945] and Lapwood [1948] were the first to analyze the onset condition of convective instability in a horizontal Darcian-porous layer for a destabilizing temperature gradient. In their works the critical value of the Darcy-Rayleigh number on the stationary convection was obtained by using linear stability theory under the principle of exchange of stabilities. Most of these instabilities manifested themselves as stationary cellular convections. Theoretical results were elucidated by the celebrated works of Combarnous and Bories [1975], Cheng [1978], and Katto and Masuoka [1967]. Combarnous and Bories [1975] noted that the convective motion in a stationary mode became oscillatory with increasing values of the Darcy-Rayleigh number beyond six or seven times larger values of its critical one. On that occasion, Caltagirone [1975] performed a stability analysis numerically and predicted the onset value of oscillatory convection in a confined porous cell. Horne and O'Sullivan [1974, 1978] extensively studied the characteristics of oscillatory convection in porous media in connection with the dominant circulation originating from steady multi-cellular patterns and the dependence of the Darcy-Rayleigh number on the frequency of oscillatory flow. The occurrence of oscillatory instabilities will be expected at the threshold stationary convection in a horizontal porous layer.

It is important to note that the above-mentioned studies focused on Newtonian fluids saturated in Darcian porous media. However, it is now widely realized that non-Newtonian fluids are applicable in various situations of polymer processing, and the critical conditions for their onsets of thermal convection in porous media must include the viscoelastic terms as well as the Darcy-Rayleigh number. Viscoelastic fluids like polymeric liquids are expected to show markedly different behaviors of evolving convective instabilities. As the elasticity of viscoelastic fluids allows the periodic instability to be sustained in addition to the stationary modes, viscoelastic fluids will exhibit an oscillatory convection at the threshold of stationary mode. For the typical Benard-Rayleigh convection of a homogenous fluid, Vest and Arpaci [1969] reported overstabilities of horizontal layer of Maxwellian fluid heated from below. Kolkka and Ierley [1987] extended these overstabilities into the Oldroyd-B fluid using linear stability theory. They suggested that buoyancy forces induce the periodic instability before the exchange of stabilities is kept stationary. Recently, Lee et al. [1993] analyzed the overstability of a Benard-Marangoni problem in the viscoelastic fluid layer heated from below. The objective of the present work is to address a stability analysis in an analytical manner and examine the effect of viscoelastic properties on oscillatory behavior of convective instabilities in a horizontal porous layer heated from below. This study will be promising in designing and operating many processes of non-Newtonian liquids in porous materials involving natural convection that cannot be explained for the stationary modes.

MATHEMATICAL FORMULATIONS

We consider an infinite horizontal porous layer saturated with viscoelastic fluids. The porous layer of vertical depth L is confined with two rigid boundaries. The bottom boundary is heated slowly with a constant temperature T_1 and upper boundary temperature is kept at a lower temperature T_2 with fixed $\Delta T(=T_1-T_2)$. It is assumed

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that, at quiescent state, the temperature varies linearly across the depth. When the magnitude of ΔT becomes larger than the critical one, thermal convection will set in due buoyancy forces. Upon the onset of thermal convection, the governing equations under the Boussinesq approximation are written as in a rectangular coordinate system with z pointing upward:

$$\nabla\cdot\vec{U}=0 \tag{1}$$

$$\left(\varepsilon\frac{\partial}{\partial t}+1\right)\vec{U}=-\frac{K}{\mu}\left(\lambda\frac{\partial}{\partial t}+1\right)(\nabla P+\vec{k}\rho g) \tag{2}$$

$$\left(\frac{\partial}{\partial t}+\vec{U}\cdot\nabla\right)T=\alpha\nabla^2 T \tag{3}$$

$$\rho=\rho_0(1-\beta(T-T_0)) \tag{4}$$

where $\vec{U}$, P, and T are the velocity vector, pressure and temperature which are all local volume-averaged, respectively [Slattery, 1967]. For a saturated liquid, ε, λ, μ, ρ and β denote the relaxation time, retardation time, viscosity, density and thermal expansion coefficient, respectively. And also K is the permeability of porous media, α is the effective thermal diffusivity, and g is the gravitational acceleration. The constitutive equation represents the modified Darcy equation employing the viscoelastic effects. For $\lambda>\varepsilon$, the approach of ε value to λ makes the viscoelastic liquid more and more exactly a Newtonian one. This type of constitutive equation for the Darcian porous layer was suggested by Alishaev and Mirzadjanzade [1975].

In this study we examine the stability condition with a linear basic temperature of temperature difference ΔT by employing linear stability theory. The current equations may be adjusted dimensionless by employing L, L^2/α, α/L and ΔT as the length, time, velocity and temperature scales, respectively. The linear equations for disturbances of velocity and temperature are reformed in terms of the vertical velocity disturbance w^*, the temperature disturbance θ^*, and the time τ, as follows:

$$\left(\bar{\varepsilon}\frac{\partial}{\partial\tau}+1\right)\nabla^2 w^*=Ra_D\left(\bar{\lambda}\frac{\partial}{\partial\tau}+1\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\theta^* \tag{5}$$

$$\frac{\partial\theta^*}{\partial\tau}-w^*=\nabla^2\theta^* \tag{6}$$

where Ra_D represents the Darcy-Rayleigh number. $\bar{\varepsilon}$ and $\bar{\lambda}$ are the dimensionless forms of ε and λ, respectively. These parameters, to characterize the present system, are defined as follows:

$$Ra_D=\frac{Kg\beta\Delta TL}{\alpha\nu},\quad \bar{\varepsilon}=\frac{\alpha}{L^2}\varepsilon,\quad \bar{\lambda}=\frac{\alpha}{L^2}\lambda$$

where ν is the kinematic viscosity of liquid. The dimensionless relaxation time $\bar{\lambda}$ means the usual Deborah number. The appropriate boundary conditions for (5) and (6) are given by

$$w^*=\theta^*=0 \qquad \text{at } z=0,1 \tag{7}$$

Eqs. (5) and (6) together with boundary conditions (7) simplify to a linear, fourth-order ordinary differential equation for the temperature disturbance, which can be readily analyzed.

STABILITY ANALYSIS

Disturbances are considered as the normal mode expansion of time-dependent periodic cells in a horizontal plane as follows:

$$w^*=w(z)\exp[i(a_x x+a_y y)+\sigma\tau] \tag{8}$$

$$\theta^*=\theta(z)\exp[i(a_x x+a_y y)+\sigma\tau] \tag{9}$$

where i is the imaginary number. a_x and a_y are the dimensionless wave numbers in the x-y horizontal plane. The temporal rate of change of disturbances σ can be decomposed into a real part and an imaginary one such as $\sigma=\sigma_r+i\sigma_i$. While the system with $\sigma_r<0$ is always stable, the condition for $\sigma_r>0$ means that the system will become unstable because of the temporal growth of disturbances. When σ equals 0, i.e. $\sigma_r=\sigma_i=0$, the system shows marginally stable state under the principle of exchange of stabilities. The minimum value of the Darcy-Rayleigh numbers for the marginal condition is regarded as the critical Darcy-Rayleigh number. Its value with the critical wave number for the stationary behavior of thermal convection in a horizontal porous layer is well known as

$$Ra_{D,c}=4\pi^2 \quad \text{with} \quad a_c=\pi \tag{10}$$

where a $(=\sqrt{a_x^2+a_y^2})$ is the horizontal wave number. This criterion corresponds to the critical condition for the onset of thermal convection in a horizontal porous layer saturated with Newtonian fluids.

In addition to the exchange of stabilities, however, periodic instabilities of viscoelastic liquids can take place even with $\sigma_r=0$. The stability equations for these periodic instabilities can be obtained by introducing Eqs. (8) and (9) into Eqs. (5)-(7), as follows:

$$(\bar{\varepsilon}\sigma+1)(D^2-a^2)(D^2-a^2-\sigma)\theta=Ra_D a^2(\bar{\lambda}\sigma+1)\theta \tag{11}$$

where D denote the differential operator d/dz. As $\bar{\varepsilon}$ and $\bar{\lambda}\to 0$, the foregoing equation approaches the classical Darcy-Rayleigh problem. The boundary conditions are reduced as

$$\theta=D^2\theta=0 \quad \text{at} \quad z=0,1 \tag{12}$$

Both boundaries are isothermal and free of viscous stresses in Darcy's law.

The resolution of the governing Eq. (11) with boundary conditions of Eq. (12) can be regarded as the usual eigenvalue problem involving eigen parameters of Ra_D, a, $\bar{\varepsilon}$, σ, and $\bar{\lambda}$ as parameters. In order to solve the present differential equation of (11) with boundary conditions (12), the required solution is assumed as

$$\theta=\theta_0\sin(n\pi z) \quad \text{for} \quad n=1,2,3,\ldots \tag{13}$$

where θ_0 denotes the integration constant subject to the boundary conditions. Substituting this form of solution into Eq. (11) results in the following characteristic equation:

$$(\bar{\varepsilon}\sigma+1)(n^2\pi^2+a^2)(n^2\pi^2+a^2+\sigma)-Ra_D a^2(\bar{\lambda}\sigma+1)=0 \tag{14}$$

This algebraic equation can be rearranged into the form of the 2nd order polynomials of σ as

$$\begin{aligned}&\bar{\varepsilon}(a^2+n^2\pi^2)\sigma^2+[\bar{\varepsilon}(a^2+n^2\pi^2)^2+(a^2+n^2\pi^2)-Ra_D a^2\bar{\lambda}]\sigma\\&\quad+(a^2+n^2\pi^2)^2-Ra_D a^2=0\end{aligned} \tag{15}$$

This equation is denoted symbolically as $A_1\sigma^2+A_2\sigma+A_3=0$. From the elementary theory of algebraic equations, it is apparent that a marginally oscillatory mode (i.e. $\sigma=i\sigma_i$) occurs at the following conditions:

$$A_1A_3>0,\quad A_2=0 \tag{16}$$

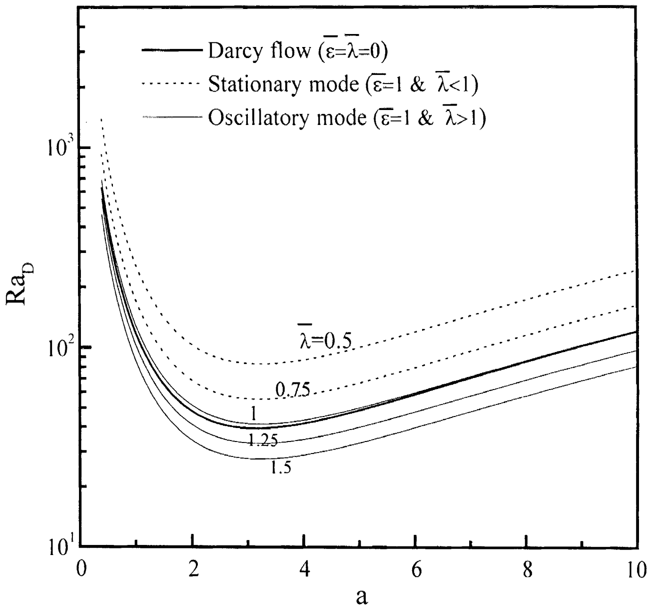

Fig. 1. Marginal stability curves for $\bar{\varepsilon}$=1.

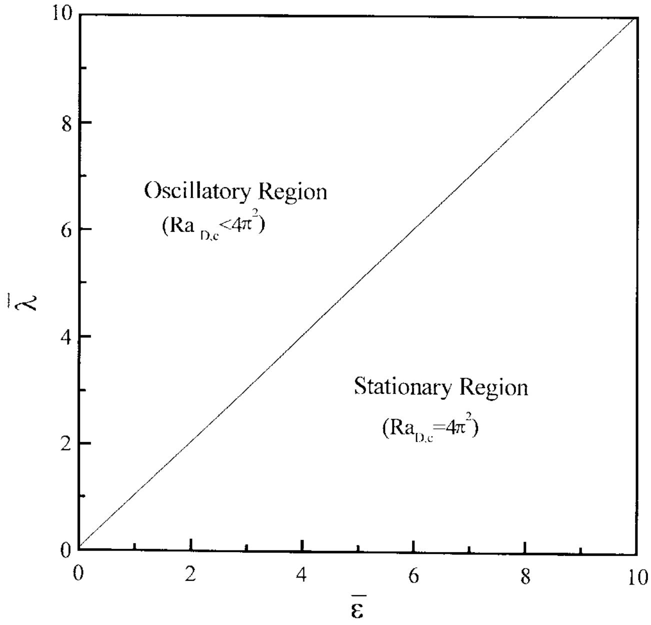

Fig. 2. Regions of overstability and stationary convection for various $\bar{\varepsilon}$ and $\bar{\lambda}$.

where the coefficients of A_i involves the eigen parameters. The first condition in the above equation implies that the critical state for the onset of stationary convection has not been attained yet, which means that stationary convection has not been initiated.

RESULTS AND DISCUSSION

With the physical meaning of the first condition of Eq. (16), the second one yields the expression for the Darcy-Rayleigh numbers as a function of eigen parameters at which marginally oscillatory modes exist:

$$Ra_D = \frac{\bar{\varepsilon}(a^2+n^2\pi^2)^2+(a^2+n^2\pi^2)}{a^2\bar{\lambda}} \tag{17}$$

The marginal stability curves for the oscillatory modes are shown in Fig. 1, where our consideration has been confined to the lowest-order mode, n=1. As the marginal stability curve for $\bar{\varepsilon}=\bar{\lambda}$ is almost the same as the one for a Darcian porous layer saturated with general Newtonian fluids, the convection will mark stationary mode in the region above the marginal stability curve for $\bar{\varepsilon}=\bar{\lambda}$. Therefore, its minimum value of critical condition equals to $Ra_{D,c}=4\pi^2$ which is for the onset of stationary convection. For a larger value of Darcy-Rayleigh number than $Ra_{D,c}=4\pi^2$, the convective motion becomes much more stable. It is impressive that all marginal stability curves for the oscillatory convection are located below the Newtonian fluid case for $\bar{\varepsilon}=\bar{\lambda}$. This means, for example, that the oscillatory convection for a porous layer saturated with a viscoelastic liquid of $\bar{\varepsilon}=1$ and $\bar{\lambda}=1.5$, is seen to start at $Ra_D=27.6$ instead of $Ra_D=4\pi^2$. It is interesting that the criteria for the overstability are dependent on the values of viscoelastic properties of saturated liquids. These criteria are shown in Fig. 2, where the characteristic curve is almost linear. It is expected that the overstability may occur for the case of larger value of $\bar{\varepsilon}$ than that of $\bar{\lambda}$. As the oscillations are first periodic in nature and then become random with a decrease in $\bar{\varepsilon}$ for a fixed $\bar{\lambda}$, an increase in $\bar{\lambda}$ for fixed $\bar{\varepsilon}$ will result in a reversal of the random pattern to a periodic oscillation.

The critical wave number, which can be obtained by minimizing Ra_D with respect to a on each curve, is given by

$$a_c^4=\pi^4+\frac{\pi^2}{\bar{\varepsilon}} \quad \text{on} \quad \frac{\partial Ra_D}{\partial a^2}=0 \tag{18}$$

The critical wave number of the oscillatory convection is always larger than that of stationary convection in a porous layer saturated with a Newtonian fluid. It is interesting to note that the wave number is independent of the relaxation time, as $Ra_D\bar{\lambda}$ is kept constant. By applying Eq. (18) into (17) for n=1, the critical Darcy-Rayleigh

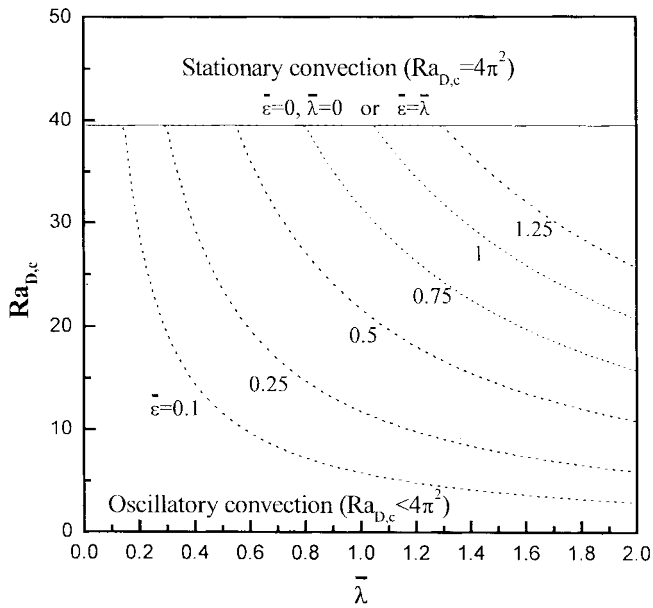

Fig. 3. Critical Darcy-Rayleigh numbers for oscillatory and stationary convections.

number for the onset of oscillatory convection is now obtained as

$$\mathrm{Ra}_{D,c}=\frac{2\bar{\varepsilon}\pi^4+2\pi^2+(2\pi^2\bar{\varepsilon}+1)\sqrt{\pi^4+\frac{\pi^2}{\bar{\varepsilon}}}}{\bar{\lambda}\sqrt{\pi^4+\frac{\pi^2}{\bar{\varepsilon}}}} \tag{19}$$

The effect of $\bar{\varepsilon}$ and $\bar{\lambda}$ on critical Darcy-Rayleigh numbers for the onset of oscillatory convection can be seen in Fig. 3. It is clear that the onset of oscillations is delayed in terms of $\mathrm{Ra}_{D,c}$ with an increase in $\bar{\varepsilon}$ and a decrease in $\bar{\lambda}$. This is due to the stabilizing effects of high $\bar{\varepsilon}$ and low $\bar{\lambda}$, which are related with viscosity and elasticity of viscoelastic fluids, respectively. This may be a result of the relative increase of buoyancy and inertia terms in Eq. (5). On the other hand, the amplitude of oscillation will decrease and the convective flow becomes stable with reduction in $\bar{\varepsilon}$.

The effect of $\bar{\varepsilon}$ and $\bar{\lambda}$ on the periods of oscillations can be investigated if we present the following expression of the dimensionless frequency for marginally oscillatory modes in terms of Eqs. (18) and (19):

$$\sigma_{i,c}=\sqrt{\frac{\pi^4+2a_c^2\pi^2+a_c^4-\mathrm{Ra}_{D,c}a_c^2}{(\pi^2+a_c^2)\bar{\varepsilon}}} \tag{20}$$

The critical value of frequency for neutrally oscillatory mode is found to be in Fig. 4 for several values of $\bar{\varepsilon}$ and $\bar{\lambda}$. As the oscillatory convection has a strong periodic nature at the onset, it is thought that the disturbances are of a single-frequency, sinusoidal character. Figure shows clearly that the frequency of oscillation decreases as the relaxation time increases. As the line for $\bar{\lambda}\rightarrow 0$ is the minimum bound for oscillatory convection, curves starting from the minimum bound line represent the properties of the frequency of oscillation. For oscillatory convection, each curve for a specific value of $\bar{\lambda}$ has its corresponding maximum value of $\bar{\varepsilon}$. But the values for the frequency of oscillation for several values of $\bar{\lambda}$ approach a specific value as $\bar{\varepsilon}\rightarrow 0$. Therefore, it is thought that the period of oscillation increases up to 330 when the Darcy-Rayleigh number increases to $\mathrm{Ra}_D=4\pi^2$. Another interesting aspect of the frequency is that the concept of Howard instability leads to periodic modulations of the depth of the thermal boundary layers responsible for the oscillation [Howard, 1964] Therefore, the thermal boundary layer is considered to play an important role in transient problems of not only stationary convection [Yoon and Choi, 1989], but oscillatory one for a horizontal porous layer heated from below. Since viscoelastic fluids exhibit markedly different stability properties as a result of possessing some elasticity, the present statement will need to be verified in relation to refined experimental work and numerical calculations.

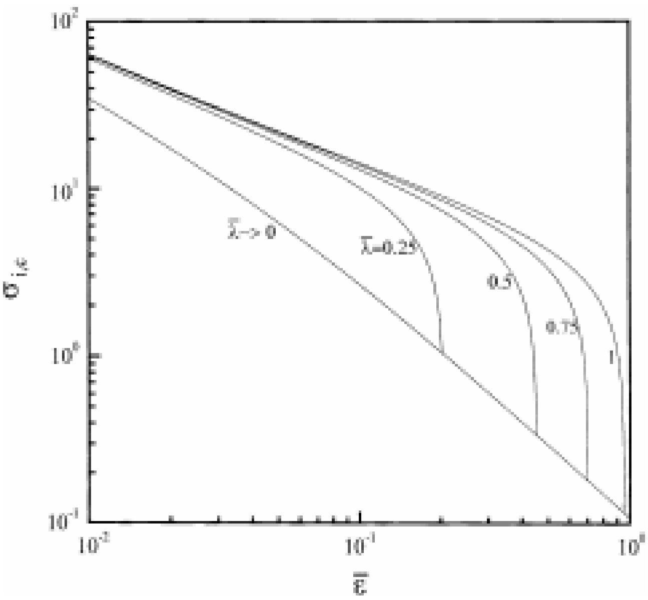

Fig. 4. Variations of critical values of frequency for neutrally oscillatory mole.

CONCLUSION

When the horizontal porous layers saturated with viscoelastic liquids are heated from below, the critical conditions to mark the onset of buoyancy-driven convection have been investigated at the convective threshold in order to exhibit an oscillatory instability. The results obtained from the linear stability theory indicate that the elasticity of saturated liquid is a destabilizing factor in the present thermal convection. Analytically, it appears that oscillatory convection may occur at a lower critical Darcy-Rayleigh number than does stationary convection. Furthermore the characteristics of the critical wave number and the frequency of oscillation have been discussed in connection with stationary convection. Even though there should be a limitation of Alisaev and Mirzadjanzade's model, this study will be helpful to understand the proper engineering situations, and also to configure systematic studies for experimental and numerical concerns.

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NOMENCLATURE

a : dimensionless horizontal wave number, $\sqrt{a_x^2+a_y^2}$
D : differential operator [d/dz]
g : gravitational acceleration [m/s^2]
K : permeability [m^2]
$\vec{k}$: unit vector in vertical direction
L : depth of porous layer [m]
P : pressure [Pa]
Ra_D : Darcy-Rayleigh number, $Kg\beta L\Delta T/\alpha\nu$
T : temperature [°C]
t : time [s]
$\vec{U}$: velocity vector [m/s]
w : dimensionless velocity component in vertical direction
x, y, z : dimensionless Cartesian coordinate

Greek Letters

α : effective thermal diffusivity [m^2/s]
β : thermal expansion coefficient [°C^{-1}]

ε : relaxation time [s]
θ : dimensionless temperature
λ : retardation time [s]
μ : dynamic viscosity [Pa·s]
ν : kinematic viscosity [m^2/s]
ρ : density [kg/m^3]
σ : dimensionless temporal growth rate
τ : dimensionless time

Subscripts
c : critical state
0 : basic state

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