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Some theoretical aspects on the onset of buoyancy-driven convection in a fluid-saturated porous medium heated impulsively from below

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Abstract−Some theoretical aspects of the onset of buoyancy-driven instability in an initially quiescent, isotropic fluid-saturated porous layer are considered. Darcy's law is employed to examine characteristics of fluid motion under the Boussinesq approximation. Using linear theory, we derive stability equations and transform them in the similarity domain. Based on linear stability equations in the similarity domain, we prove the principle of exchange of stabilities and show that the stability parameter is stationary. The temperature disturbance field is expressed as a series of orthonormal functions and the vertical velocity one is obtained in simple recursive form. The validity of the quasi-steady state approximation (QSSA) is also proved by comparing the stability characteristics under the QSSA with those obtained from the eigenanalysis without the QSSA.

Keywords: Buoyancy-driven Instability, Porous Media, Exchange of Stabilities, Eigenanalysis, Propagation Theory

INTRODUCTION

The analysis of convective instabilities in the horizontal porous layer begins with Horton-Rogers-Lapwood convection [1,2]. They examined thermally-driven convection by using methods developed for convection in a homogeneous fluid [3]. In their work temperature increases linearly with depth, which is appropriate for gradual heating or for a steady state, e.g., naturally occurring geothermal gradients in the subsurface. However, in many experimental situations and field studies a relatively rapid change in temperature or solute concentration occurs at one boundary. The basic profile of temperature or concentration before the onset of convection develops with time. Under linear stability theory, Caltagirone [4] first tried to analyze this transient Horton-Rogers-Lapwood convection using the quasi-steady state approximation and initial value problem approach. The energy method is also considered in Caltagirone's work [4]. Later, Yoon and Choi [5] reformulated the linear stability equation in the similarity domain by adopting the thermal penetration depth as a proper length scaling factor and calculated the stability conditions.

Recently, carbon dioxide sequestration in deep saline aquifers has been considered as one of the most feasible long-term CO_2 storage strategies. In this CO_2 geological sequestration process, the supercritical CO_2 is injected into geologically stable formations, and it gradually dissolves into subsurface brine saturated in the porous formations. Unlike other atmospheric gases, the dissolved CO_2 increases the density of brine, makes the fluid system gravitationally unstable, and induces convective motion. To analyze the onset of convective instability in the CO_2 geological sequestration process, Ennis-King et al. [6], Xu et al. [7] and Hassanzadeh et al. [8] corrected Caltagirone's analysis [4] and extended it into anisotropic porous media. They conducted their analysis in the global domain. Later, Riaz et al. [9], Selim and Rees [10], Wessel-Berg [11] and Kim and Choi [12] reexamined the above problem under the transient concentration or temperature field in the similarity domain. Even though the above studies give good understanding on the onset of buoyancy-driven convection in a porous medium, some theoretical issues remain to be clarified.

In the present study, we consider some theoretical aspects on the buoyancy-driven instability in an isotropic porous medium. Based on the linear stability equations in the similarity domain, we prove the principles of exchange of stabilities and show that the characteristic stability parameter has a stationary property. Based on the above theoretical basis, we examine the validity of the QSSA which has been employed in the present and similar systems.

GOVERNING EQUATIONS

The system considered here is an initially quiescent, fluid-saturated, horizontal porous layer of depth d, as shown in Fig. 1. The porous layer has a constant porosity, ε, and a constant permeability, K. The interstitial fluid is characterized by the thermal expansion coefficient, β, density, ρ, heat capacity, $(\rho c)_f$ and kinematic viscosity, ν. The porous medium is regarded as a homogeneous and isotropic material with heat capacity $(\rho c)_e=\varepsilon(\rho c)_f+(1-\varepsilon)(\rho c)_m$ and thermal conductivity $k_e=\varepsilon k_f+(1-\varepsilon)k_m$. Here subscripts f and m stand for fluid and porous matrix, respectively. Before heating, the fluid layer is maintained at a uniform temperature, T_i. For time $t\geq 0$ the lower boundary is suddenly heated with a constant temperature, T_b. For this system the governing equations of flow and temperature fields are expressed by employing the Boussinesq approximation and Darcy's model [10]:

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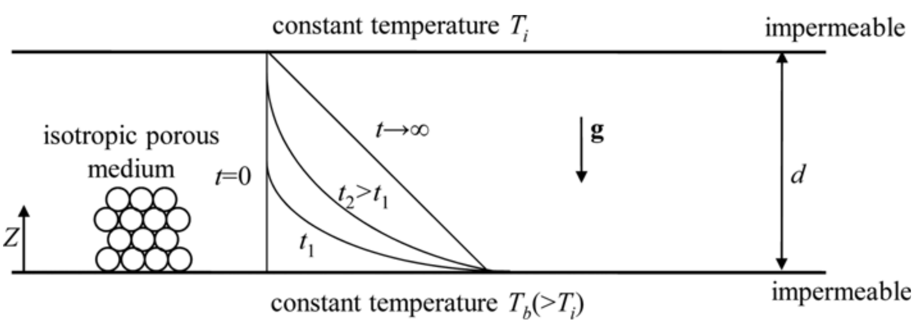

Fig. 1. Schematic diagram of the system considered here.

$$\nabla\cdot\mathbf{U}=0, \tag{1}$$

$$\frac{\mu}{K}\mathbf{U}=-\nabla P+\rho\mathbf{g}, \tag{2}$$

$$\left(\frac{\partial}{\partial t}+\frac{(\rho c)_f}{(\rho c)_e}\mathbf{U}\cdot\nabla\right)T=\alpha\nabla^2 T, \tag{3}$$

$$\rho=\rho_i[1-\beta(T-T_i)], \tag{4}$$

where **U** is the velocity vector, T the temperature, P the pressure, μ the viscosity, $\alpha(=k_e/(\rho c)_e)$ the effective thermal diffusivity, ρ_i the reference density and **g** the gravitational acceleration. Here the local thermal equilibrium between the fluid and the porous matrix phase is assumed. The important parameter to describe the present system is the Darcy-Rayleigh number Ra_D defined by

$$Ra_D=\frac{g\beta K\Delta T d}{\alpha\nu}\frac{(\rho c)_f}{(\rho c)_e}, \tag{5}$$

where $\Delta T=T_b-T_i$.

A set of nondimensionalized variables z, τ and θ_0 is defined by using the scale of vertical length d, time d^2/α, and temperature ΔT. Then the basic conduction state is represented in dimensionless form by

$$\frac{\partial\theta_0}{\partial\tau}=\frac{\partial^2\theta_0}{\partial z^2} \tag{6}$$

with the following initial and boundary conditions,

$$\theta_0=0 \text{ at } \tau=0, \tag{7a}$$
$$\theta_0=1 \text{ at } z=0 \text{ and } \theta_0=0 \text{ at } z=1. \tag{7b}$$

The above equations can be solved by using the conventional separation of variables technique or Laplace transform as follows:

$$\theta_0=1-z-2\sum_{n=0}^{\infty}\frac{1}{n\pi}\sin(n\pi z)\exp(-n^2\pi^2\tau), \tag{8a}$$

$$\theta_0=\sum_{n=0}^{\infty}\left\{\operatorname{erfc}\left(\frac{n}{\sqrt{\tau}}+\frac{\zeta}{2}\right)-\operatorname{erfc}\left(\frac{n+1}{\sqrt{\tau}}-\frac{\zeta}{2}\right)\right\}, \tag{8b}$$

where $\zeta=z/\sqrt{\tau}$. Eq. (8b), which is based on the boundary-layer (τ, ζ) coordinates rather than the global (τ, z) ones, converges more rapidly than Eq. (8a) for a small-time region. The evolution of the basic profiles of temperature with time is described in Fig. 2. For the deep-pool system of $\tau\le 0.01$ the basic temperature profiles are reduced to

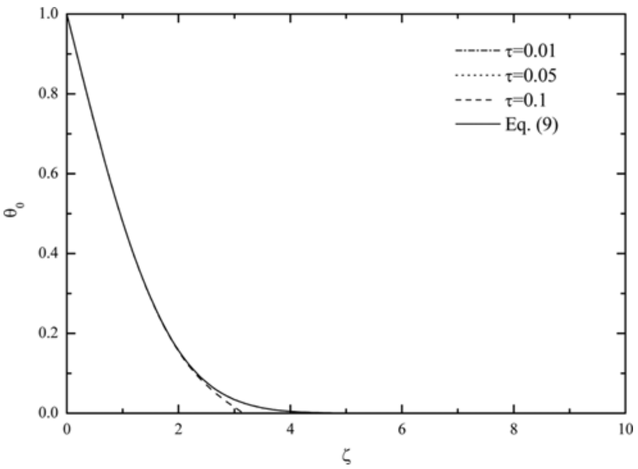

Fig. 2. Basic temperature fields. For τ>0.1, the basic temperature fields clearly deviate from the similarity solution (9).

$$\theta_0=\operatorname{erfc}\left(\frac{\zeta}{2}\right), \tag{9}$$

which is self-similar. The Leveque-type solution (9) is in good agreement with the exact solution (8) in the region of $\tau<0.1$.

THEORETICAL ANALYSIS

1. Linear Stability Theory

Under linear stability theory the disturbances caused by the onset of thermal convection can be formulated, in dimensionless form, in terms of the temperature component θ_1 and the vertical velocity component w_1 by decomposing Eqs. (1)-(4):

$$\nabla^2 w_1=-\nabla_1^2\theta_1, \tag{10}$$

$$\frac{\partial\theta_1}{\partial\tau}+Ra_D w_1\frac{\partial\theta_0}{\partial z}=\nabla^2\theta_1, \tag{11}$$

where $\nabla^2=(\partial^2/\partial x^2)+(\partial^2/\partial y^2)+(\partial^2/\partial z^2)$ and $\nabla_1^2=(\partial^2/\partial x^2)+(\partial^2/\partial y^2)$. Here the temperature and velocity components have the scale of $\Delta T/Ra_D$ and α/d, respectively. The proper boundary conditions are given by

$$w_1=\theta_1=0 \text{ at } z=0 \text{ and } z=1. \tag{12}$$

The boundary conditions represent no flow through the boundaries and the fixed temperature on the boundaries.

According to the Fourier mode analysis, convective motion is assumed to exhibit the horizontal periodicity [3]. Then the perturbed quantities can be expressed as follows:

$$[w_1(\tau, x, y, z), \theta_1(\tau, x, y, z)]=[w_1(\tau, z), \theta_1(\tau, z)]\exp[i(k_x x+k_y y)], \tag{13}$$

where "i" is the imaginary number. Substituting the above Eq. (13) into Eqs. (10) and (11) produces the usual amplitude functions in terms of the horizontal wavenumber $k=(k_x^2+k_y^2)^{1/2}$:

$$\left(\frac{\partial^2}{\partial z^2}-k^2\right)w_1=k^2\theta_1, \tag{14}$$

$$\frac{\partial\theta_1}{\partial\tau}=\left(\frac{\partial^2}{\partial z^2}-k^2\right)\theta_1-Ra_D w_1\frac{\partial\theta_0}{\partial z}. \tag{15}$$

For the extreme case of $\sqrt{\tau}<<1$, it is expected that the onset time is independent of the vertical extent of the system. In other words, the system can be treated in a semi-infinite domain. For this case, as discussed by Ben et al. [13], Riaz et al. [9] and Pritchard [14], the dominant diffusional operator, $\partial^2/\partial z^2$ does not have localized eigenfunctions that vanish at the infinite boundary, and therefore, the disturbances which are localized near the diffusion front cannot be accurately captured in the (τ, z)-domain. Following a coordinate transformation to the similarity variable of the base state $\zeta=z/\sqrt{\tau}$, the disturbance equations can be expressed as

$$\left(\frac{1}{\tau}\frac{\partial^2}{\partial\zeta^2}-k^2\right)w_1=k^2\theta_1, \tag{16}$$

$$\frac{\partial\theta_1}{\partial\tau}+Ra_D\frac{1}{\sqrt{\tau}}w_1\frac{\partial\theta_0}{\partial\zeta}=\frac{1}{\tau}\left(\frac{\partial^2}{\partial\zeta^2}+\frac{\zeta}{2}\frac{\partial}{\partial\zeta}-k^2\tau\right)\theta_1, \tag{17}$$

under the following boundary conditions:

$$w_1=\theta_1=0 \text{ at } \zeta=0, \tag{18a}$$
$$w_1\to 0 \text{ and } \theta_1\to 0 \text{ as } \zeta\to\infty. \tag{18b}$$

Note that the transformation to the semi-infinite domain by $(\tau, z)\to(\tau, \zeta)$ is singular at $\tau=0$. Hence, the temporal evolution of the disturbance must be restricted away from the singular limit of $\tau=0$. In any case, any semi-infinite domain solution produces the unrealistic limit of the infinite gradient or amplitude at $\tau=0$.

2. Exchange of Stabilities

Now, we shall show that the principle of the exchange of stabilities is valid. To prove the exchange of stabilities, let us rearrange Eq. (17) as

$$\left(D^2+\frac{\zeta}{2}D-k^{*2}-\tau\frac{\partial}{\partial\tau}\right)\theta_1=-Ra_D^* w_1\frac{1}{\sqrt{\pi}}\exp\left(-\frac{\zeta^2}{4}\right). \tag{19}$$

where $D=\partial/\partial\zeta$ and $k^*(=k\sqrt{\tau})$ is the wavenumber rescaled with the penetration depth $\Delta(=d\sqrt{\tau})$. By multiplying Eq. (19) by $\exp(\zeta^2/4)\bar{\theta}_1$ and integrating over the range of ζ, the following relation can be obtained;

$$\tau\left\{\int_0^\infty\exp\left(\frac{\zeta^2}{4}\right)\bar{\theta}_1\frac{\partial\theta_1}{\partial\tau}d\zeta\right\}-\int_0^\infty\exp\left(\frac{\zeta^2}{4}\right)\bar{\theta}\left(D^2+\frac{\zeta}{2}D-a^{*2}\right)\theta_1 d\zeta$$
$$=\frac{1}{\sqrt{\pi}}Ra_D^*\int_0^\infty w_1\bar{\theta}_1 d\zeta. \tag{20}$$

where $\bar{\theta}_1$ is the complex conjugate of θ_1 and $\exp(\zeta^2/4)$ is the weighting function of the following Sturm-Liouville equation:

$$L_\zeta\phi_n=-\lambda_n\phi_n, \tag{21}$$

where $L_\zeta=D^2+(\zeta/2)D$. Eigenfunctions and eigenvectors for Eq. (21) are to be discussed later.

By integration by parts we obtain

$$\int_0^\infty\exp\left(\frac{\zeta^2}{4}\right)\bar{\theta}_1\left(D^2+\frac{\zeta}{2}D-k^{*2}\right)\theta_1 d\zeta=-\int_0^\infty\exp\left(\frac{\zeta^2}{4}\right)|D\theta_1|^2 d\zeta$$
$$-\int_0^\infty\exp\left(\frac{\zeta^2}{4}\right)k^{*2}|\theta_1|^2 d\zeta, \tag{22}$$

since the integrated part disappears through the boundary condition (3.9). By similar approach, the integration of the right side of Eq. (20) yields

$$\int_0^\infty\bar{\theta}_1 w_1 d\zeta=-\frac{1}{k^{*2}}\int_0^\infty w_1(D^2-k^{*2})\bar{w}_1 d\zeta=\frac{1}{k^{*2}}\int_0^\infty|Dw_1|^2+k^{*2}|w_1|^2 d\zeta. \tag{23}$$

Combining Eqs. (20), (22) and (23), the following relation is obtained:

$$\sigma^*\tau\int_0^\infty\exp\left(\frac{\zeta^2}{4}\right)|\theta_1|^2 d\zeta+\int_0^\infty\exp\left(\frac{\zeta^2}{4}\right)\{|D\theta_1|^2+k^{*2}|\theta_1|^2\}d\zeta$$
$$-\frac{Ra_D^*}{k^{*2}}\int_0^\infty|Dw_1|^2+k^{*2}|w_1|^2 d\zeta=0. \tag{24}$$

where the temporal growth rate, σ^* is defined as

$$\sigma^*=\frac{1}{\int_0^\infty|\theta_1|^2\exp\left(\frac{\zeta^2}{4}\right)d\zeta}\left\{\int_0^\infty\frac{\partial\theta_1}{\partial\tau}\bar{\theta}_1\exp\left(\frac{\zeta^2}{4}\right)d\zeta\right\}, \tag{25}$$

is a function of τ only, for a given k. The real and imaginary part of the this equation must vanish separately; therefore, the imaginary part should be

$$Im(\sigma^*)\tau\int_0^\infty\exp\left(\frac{\zeta^2}{4}\right)|\theta_1|^2 d\zeta=0. \tag{26}$$

Since the integration $\int_0^\infty\exp\left(\frac{\zeta^2}{4}\right)|\theta_1|^2 d\zeta$ has a real value,

$$Im(\sigma^*)=0. \tag{27}$$

This means that σ^* is a real function for all positive τ and that the principle of the exchange of stabilities is valid for the present problem. Since $\sigma^*=\bar{\sigma}^*$, the temporal growth rate defined in Eq. (25) can be rewritten as

$$\sigma^*=\frac{1}{2}\frac{1}{\int_0^\infty|\theta_1|^2\exp\left(\frac{\zeta^2}{4}\right)d\zeta}\left\{\int_0^\infty\left(\frac{\partial\theta_1}{\partial\tau}\bar{\theta}_1+\theta_1\frac{\partial\bar{\theta}_1}{\partial\tau}\right)\exp\left(\frac{\zeta^2}{4}\right)d\zeta\right\} \tag{28}$$

3. Variational Principles

The transition from stability to instability must precede the stationary state since σ^* is a real function for all positive τ. By setting

$\sigma^*=0$, the neutral state is characterized from Eq. (24) as

$$\mathrm{Ra}_D^*=\frac{\mathrm{k}^{*2}\sqrt{\pi}\int_0^\infty \exp\left(\frac{\zeta^2}{4}\right)\{|\mathrm{D}\theta_1|^2+\mathrm{k}^{*2}|\theta_1|^2\}\mathrm{d}\zeta}{\int_0^\infty |\mathrm{Dw}_1|^2+\mathrm{k}^{*2}|\mathrm{w}_1|^2\mathrm{d}\zeta}=\frac{\mathrm{I}_1}{\mathrm{I}_2}. \tag{29}$$

Let $\delta\mathrm{Ra}_D^*$ be the change in Ra_D^* when w_1 is subjected to a small variation $\delta\mathrm{w}_1$, which satisfies the boundary condition on w_1. From Eq. (29) the following variational relation is obtained:

$$\delta\mathrm{Ra}_D^*=\frac{1}{\mathrm{I}_2}(\delta\mathrm{I}_1-\mathrm{Ra}_D^*\delta\mathrm{I}_2), \tag{30}$$

where

$$\delta\mathrm{I}_1=2\mathrm{k}^{*2}\sqrt{\pi}\int_0^\infty \exp\left(\frac{\zeta^2}{4}\right)\{\mathrm{D}\theta_1\delta(\mathrm{D}\theta_1)+\mathrm{k}^{*2}\theta_1\delta\theta_1\}\mathrm{d}\zeta, \tag{31}$$

and

$$\delta\mathrm{I}_2=2\int_0^\infty \{\mathrm{Dw}_1\delta(\mathrm{Dw}_1)+\mathrm{k}^{*2}\mathrm{w}_1\delta\mathrm{w}_1\}\mathrm{d}\zeta. \tag{32}$$

After a sequence of integrations by parts, the following relation is obtained

$$\delta\mathrm{I}_1=2\mathrm{k}^*\sqrt{\pi}\int_0^\infty \exp\left(\frac{\zeta^2}{4}\right)\left\{\mathrm{D}^2\theta_1+\frac{\zeta}{2}\mathrm{D}\theta_1+\mathrm{k}^{*2}\theta_1\right\}\delta\theta_1\mathrm{d}\zeta, \tag{33}$$

and

$$\delta\mathrm{I}_2=2\mathrm{k}^{*2}\int_0^\infty \mathrm{w}_1\delta\theta_1\mathrm{d}\zeta. \tag{34}$$

Therefore,

$$\delta\mathrm{Ra}_D^*=\frac{2\mathrm{k}^{*2}}{\mathrm{I}_2}\sqrt{\pi}\int_0^\infty \exp\left(\frac{\zeta^2}{4}\right)\left\{\left(\mathrm{D}^2+\frac{\zeta}{2}\mathrm{D}-\mathrm{k}^{*2}\right)\theta_1 +\mathrm{Ra}_D^*\mathrm{w}_1\frac{1}{\sqrt{\pi}}\exp\left(-\frac{\zeta^2}{4}\right)\right\}\delta\theta_1\mathrm{d}\zeta, \tag{35}$$

From the above equation, it follows that $\delta\mathrm{Ra}_D^*=0$ if

$$\left(\mathrm{D}^2+\frac{\zeta}{2}\mathrm{D}-\mathrm{k}^{*2}\right)\theta_1=-\mathrm{Ra}_D^*\mathrm{w}_1\frac{1}{\sqrt{\pi}}\exp\left(-\frac{\zeta^2}{4}\right). \tag{36}$$

The above argument means that the characteristic value Ra_D^* has a stationary property when Eq. (36) is evaluated in terms of characteristic functions which satisfy Eqs. (26)-(28).

4. Eigen Anlaysis and Propagation Theory

Recently, Kim and Choi [12] expressed the disturbance fields as

$$\theta_1=\sum_{n=1}^\infty \mathrm{a}_n(\tau)\alpha_n\phi_n(\zeta) \tag{37}$$

Here the eigenfunction ϕ_n and corresponding eigenvalue λ_n of the Sturm-Liouville Eq. (21) are

$$\phi_n=\mathrm{H}_{2n-1}\left(\frac{\zeta}{2}\right)\exp\left(-\frac{\zeta^2}{4}\right) \text{ and } \lambda_n=\mathrm{n}=1, 2, \cdots \tag{38a and b}$$

where H_k is the k-th Hermite polynomial. The scale factor $\alpha_n=\{2^{n-1/2}\pi^{1/4}\sqrt{\Gamma(2\mathrm{n})}\}^{-1}$ is inserted to guarantee the orthonormal relation:

$$\alpha_m\alpha_n\int_0^\infty \phi_m\phi_n\exp\left(\frac{\zeta^2}{4}\right)\mathrm{d}\zeta=\delta_{mn}, \tag{39}$$

where $\exp(\zeta^2/4)$ is the weighting function of the Sturm-Liouville Eq. (21).

From Eqs. (16), (18) and (37), w_1 is expressed as

$$\mathrm{w}_1=\sum_{n=1}^\infty \mathrm{a}_n(\tau)\alpha_n\psi_n(\mathrm{k}^*, \zeta), \tag{40}$$

where ψ_n can be obtained by solving

$$(\mathrm{D}^2-\mathrm{k}^{*2})\psi_n=\mathrm{k}^{*2}\phi_n, \tag{41a}$$

under the following boundary conditions:

$$\psi_n=0 \text{ at } \zeta=0 \text{ and } \zeta\to\infty. \tag{41b}$$

Using the method of variation of parameters [11] or the inverse operator technique [12], the solutions of Eq. (41) can be expressed as

$$\psi_n=-\mathrm{k}^*\sinh(\mathrm{k}^*\zeta)\int_0^\infty \phi_n\exp(-\mathrm{k}^*\xi)\mathrm{d}\xi-\frac{\mathrm{k}^*}{2}\exp(-\mathrm{k}^*\zeta)\int_0^\zeta \phi_n\exp(\mathrm{k}^*\xi)\mathrm{d}\xi +\frac{\mathrm{k}^*}{2}\exp(\mathrm{k}^*\zeta)\int_0^\zeta \phi_n\exp(-\mathrm{k}^*\xi)\mathrm{d}\xi. \tag{42}$$

After performing the integrations, Eq. (42) can be simplified recursively as

$$\psi_n=4\mathrm{k}^{*2}(\psi_{n-1}-\phi_{n-1}),\ \ \mathrm{n}=2, 3, \cdots \tag{43a}$$

with

$$\psi_1=-\mathrm{k}^{*2}\sqrt{\pi}\exp(\mathrm{k}^{*2})\left[\exp(-\mathrm{k}^*\zeta)\mathrm{erf}\left(-\mathrm{k}^*+\frac{\zeta}{2}\right) +\exp(\mathrm{k}^*\zeta)\mathrm{erf}\left(\mathrm{k}^*+\frac{\zeta}{2}\right)-2\sinh(\mathrm{k}^*\zeta)\right]. \tag{43b}$$

Substituting θ_1 and w_1 into Eq. (17) and performing the orthogonalization process, the stability equations are reduced to the following matrix form:

$$\tau\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}\tau}=\mathbf{Ba}, \tag{44a}$$

where

$$\mathrm{B}_{mn}=-(\lambda_m+\mathrm{k}^{*2})\delta_{nm}+\mathrm{Ra}_D^*\alpha_m\alpha_n\mathrm{h}_{mn}, \tag{44b}$$

$$\mathrm{h}_{mn}(\mathrm{k}^*)=\int_0^\infty \phi_m(\zeta)\psi_n(\mathrm{k}^*, \zeta)\mathrm{d}\zeta, \tag{44c}$$

$$\mathbf{a}=[\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \cdots]^T, \tag{44d}$$

for m, n=1, 2, ⋯. It is stressed that the partial differential Eqs. (16)-(18) are reduced to simultaneous ordinary differential Eqs. (44), without spatial discretization. Furthermore, the characteristic matrix $\mathbf{B}$ is normal, $\mathbf{B}=\mathbf{B}^T$, since $\mathrm{h}_{mn}=\mathrm{h}_{nm}$ through

$$\mathrm{h}_{mn}=\int_0^\infty \psi_m\phi_n \mathrm{d}\zeta=-\frac{1}{\mathrm{k}^{*2}}\int_0^\infty \psi_m\left(\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}-\mathrm{k}^{*2}\right)\psi_n \mathrm{d}\zeta$$
$$=\int_0^\infty\left(\frac{1}{\mathrm{k}^{*2}}\frac{\mathrm{d}\psi_m}{\mathrm{d}\zeta}\frac{\mathrm{d}\psi_n}{\mathrm{d}\zeta}+\psi_m\psi_n\right)\mathrm{d}\zeta=\mathrm{h}_{nm}. \quad (45)$$

To trace the growth of the disturbance, the norm of the disturbance $\|\theta_1\|$ is defined as

$$\|\theta_1\|=\left[\int_0^\infty \theta_1^2\exp\left(\frac{\zeta^2}{4}\right)\mathrm{d}\zeta\right]^{1/2}=\sqrt{\mathbf{a}^H\mathbf{a}}, \quad (46)$$

Based on the this quantity, one can define the growth rate σ^* as

$$\sigma^*=\frac{1}{\|\theta_1\|}\frac{\mathrm{d}\|\theta_1\|}{\mathrm{d}\tau}=\frac{1}{2\mathbf{a}^H\mathbf{a}}\left(\frac{\mathrm{d}\mathbf{a}^H}{\mathrm{d}\tau}\mathbf{a}+\mathbf{a}^H\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}\tau}\right), \quad (47)$$

which is equivalent to Eq. (25). With the aid of basic matrix operation, the growth rate defined in Eq. (47) can be rewritten as

$$\sigma^*\tau=\frac{\mathbf{a}^H\mathbf{E}\mathbf{a}}{\mathbf{a}^H\mathbf{a}}, \quad (48)$$

where $\mathbf{E}=(\mathbf{B}^H+\mathbf{B})/2$. Note that $\mathbf{E}=\mathbf{B}$ since the matrix **B** is normal.

For the normal matrix **B**, the following Rayleigh quotient can be defined [15]:

$$\mathrm{R}(\mathbf{B},\mathbf{a})=\frac{\mathbf{a}^H\mathbf{B}\mathbf{a}}{\mathbf{a}^H\mathbf{a}}. \quad (49)$$

And, the maximum value of R(**B**, **a**) reaches the maximum eigenvalue of **B** when **a** is the corresponding eigenvector:

$$\mathrm{R}(\mathbf{B},\mathbf{a})\le\max\{\mathrm{eig}(\mathbf{B})\}. \quad (50)$$

By using Eqs. (48) and (50), the following relation is obtained:

$$\sigma^*\tau\le\max\{\mathrm{eig}(\mathbf{B})\} \text{ at } \tau=\tau \quad (51)$$

Therefore, the growth rate of the most dangerous disturbances at a given time τ defined in Eq. (50) can be expressed as

$$\sigma^*\tau=\max\{\mathrm{eig}(\mathbf{B})\}. \quad (52)$$

Yoon and Choi [5] reformulated the linear stability equation by adopting the thermal penetration depth as a proper length and calculated the stability conditions. They called their method propagation theory. Under propagation theory, they assumed $\mathrm{d}\mathbf{a}/\mathrm{d}\tau=\sigma^*\mathbf{a}$ and expressed the dimensionless amplitude functions of disturbances as

$$[\mathrm{w}_1(\tau,\zeta),\theta_1(\tau,\zeta)]=[\mathrm{w}^*(\mathrm{k}^*,\zeta),\theta^*(\zeta)]\exp(\sigma^*\tau). \quad (53)$$

Then, the stability equation is obtained from the Eqs. (16)-(18) as

$$(\mathrm{D}^2-\mathrm{k}^{*2})\mathrm{w}^*=\mathrm{k}^{*2}\theta^*, \quad (54)$$

$$\sigma^*\tau\theta^*+\mathrm{Ra}_D^*\mathrm{w}^*\mathrm{D}\theta_0=\left(\mathrm{D}+\frac{\zeta}{2}\mathrm{D}-\mathrm{k}^{*2}\right)\theta^*, \quad (55)$$

$$\mathrm{w}^*=\theta^*=0 \text{ at } \zeta=0, \quad (56a)$$
$$\mathrm{w}^*\to0 \text{ and } \theta^*\to0 \text{ as } \zeta\to\infty. \quad (56b)$$

Selim and Rees [10] derived the above stability equation under the quasi-steady state approximation (QSSA). This means that Yoon and Choi's [5] propagation theory corresponds to the QSSA in the (τ,ζ)-domain. Setting $\sigma^*=0$ in Eq. (55), and approximating θ_0 as a simple fourth-order polynomial, Yoon and Choi [5] suggested the critical conditions as

$$\tau_c=154.5\mathrm{Ra}_D^{-2} \text{ and } \mathrm{k}_c=0.0736\mathrm{Ra}_D.$$

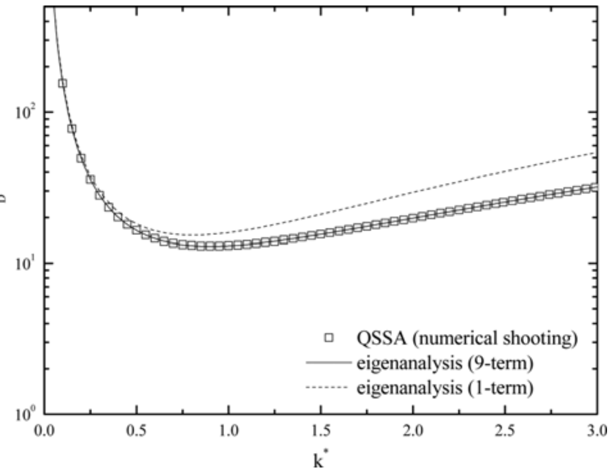

Fig. 3. Comparison of the neutral stability curves obtained from the QSSA and the present eigen analysis.

Recently, Kim and Choi [12] showed that the present analytical eigen analysis and the QSSA of Selim and Rees's [10] give exactly the same stability conditions. The neutral stability curves obtained from the numerical shooting method [16] based on the QSSA and the present eigen analysis using nine terms in Eqs. (37) and (40) are compared in Fig. 3. The leading-term analysis which corresponds to the 1-term approximation is also conducted and summarized in Fig. 3. This figure implies that the present QSSA, Eq. (55) is useful for the present system whose characteristic matrix **B** is normal. The basic reason for the present consistency might arise from the fact that the base field is time-independent in the (τ,ζ)-domain, as shown in Eq. (9). Therefore, Yoon and Choi's [5] propagation theory is a good approximation of the present exact the eigen analysis.

CONCLUSIONS

Some analytical features of the Horton-Rogers-Lapwood convection under the transient base field have been considered here. We proved analytically the principle of exchange of stabilities and showed that the characteristic stability parameter Ra_D^* has a stationary property. Furthermore, we suggested disturbance fields which are more compact than the previous ones. Also, we found that the characteristic matrix **B** is normal, and therefore, the growth rate can be obtained from the maximum eigenvalue of the matrix **B**. It is very interesting that the present stability results are exactly same as those under the QSSA and the propagation theory Based on these, the present study might give a robust theoretical background, which has been assumed and ignored in the previous studies. Furthermore, considering the similarity between heat transfer and mass transfer systems, the present study can be applied to the onset of buoyancy-driven convection in the CO_2 geological sequestration process where convective motion accelerates the CO_2 dissolution into the brine saturated in geologically stable formations.

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