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NONLINEAR DYNAMICS AND STRANGE ATTRACTORS

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Abstract—Stochastic self-oscillator is one of the most intriguing features in nonlinear dynamics, and usually can be observed at the accumulation point of successive period doubling bifurcations. In regard to this problem, the substance deals with the stability and bifurcation aspects of oscillatory motions, fine structure of trajectories, typical properties shown in the flows and maps, and the mathematical rigor in measuring the stochasticity concerned with applications to science and engineering problems.

INTRODUCTION

The problems arising in many contexts of engineerings and natural sciences are frequently interpreted in the form of mathematical equations through modelling techniques. A dynamical system, which may be thought of as any set of equations giving the time evolution of the state of a system from a knowledge of its previous history, can have the form of a function (though this term is frequently replaced by the word mapping in dynamics), a set of first order ordinary differential equations or of partial differential equations. Such a system, depending on the mathematical aspects of its structure, can exhibit a variety of dynamic behavior from stable fixed points to a hierarchy of stable limit cycles or apparently stochastic oscillations.

Stochastic oscillations or chaos, involve an attractive random set (more usually called the strange attractor) within which all the paths in the phase space of the dynamical system are unstable and behave in a complicated and tortuous fashion. These stochastic oscillations are intimately related with the problem of turbulence, which arose almost a hundred years ago and has remained to this day one of the most attractive and intriguing problem in classical physics and is still far away from its final solution. The problem of turbulence, which originally appeared in hydrodynamics, is in fact common to many branches of science such as plasma physics, cosmology, ecology, weather forcasting, the theory of planets and stars, chemical kinetics, radiophysics and so on.

Early in 1940s, Landau[1] suggested that the onset of turbulence in fluid flow may be viewed as sequential instabilities of the state. An analogous idea was put forward independently by Hopf[2] in a somewhat different form. The Landau-Hopf model suggested that as the Reynold number increases the turbulence appears as a result of a chain of successive bifurcations that leads to a quasiperiodic motion. The first bifurcation in this chain is such that the initially stable state of equilibrium is transformed into an unstable state and, at the same time, a stable limit cycle appears in its neighborhood. The resulting periodic motion then loses its stability, and a two dimensional formation appears in the neighborhood of the stable cycle that has vanished, namely, a torus whose winding frequency is unrelated to the main frequency. This doubly periodic motion then becomes unstable and a three dimensional torus is created, and so on. The result of such sequential bifurcations is that the motion becomes very complicated and tortuous.

During the 1960s, Lorenz[3] and Ruelle and Takens[4] independently and from different points of view, suggested the relevance of strange attractors to the onset of turbulence. Lorenz wanted to explain the dynamics of a model system of three coupled, first order, nonlinear evolution of the Benard instability. By a careful analysis of the numerical solutions, he discovered an exotic solution which wandered in a region of the phase space of the system with very complicated geometrical structure. Ruelle and Takens too offered a possible mechanism to a turbulent solution. They discussed, on the basis of general arguments, the strange attractor that could appear via transition from a doubly periodic motion on a two dimensional toroidal surface.

Of chemically reacting systems the most thoroughly studied oscillating system is the Belousov-Zhabotinskii reaction[5-12] which involves the ceriumcatalyzed bromination and oxidation of malonic acid by a sulfuric acid solution of bromate. The earlier model of B-Z system involved reactions among eleven chemical species and some of the proposed systems [11,12] are of questionable chemical relevance even though they display a variety of mathematically interesting behavior. Recent models of complex dynamics are the three-dimensional systems of consecutive reactions A→B→C in a continuous flow stirred tank reactor[13,14], which display sequential bifurcations of period doubling and chaotic motions.

Besides the literature above mentioned, there has been much published in this field. The problems under discussion involve the fluid flows[15,16], solid state physics[17], buckling beams[18,19], quantum mechanics[20,21], plasmas[22,23] and magnetohydrodynamic flows[24,25]. It is not possible to give a detailed presentation of all of these and for more information, the references should be consulted.

NONLINEAR DYNAMICS

1. Nonlinear Systems and Maps

In looking into the nature of stochastic nonlinear dynamics, we first review some aspects of the initial value problem

$$dx/dt = f(x), \quad x(0) = x_0 \tag{1}$$

where x is a state vector in n-dimensional real space R, as may be expressed by $x \epsilon R^n$. Then, the flow ϕ_t: $R^n \rightarrow R^n$ is defined by $\phi_t(x_o) = x(t,x_o)$ and is read as the map ϕ_t from R^n into itself. The equilibria of the system are referred to as the zeroes of f or the fixed points of (1). Suppose that we have fixed point $\bar{x}$ so that $f(\bar{x}) = 0$. Then, to characterize the behavior of the solution near $\bar{x}$, we usually use the linearization technique and know that there exist, in the linearized system, stable and unstable eigenspaces E^s, E^u of dimension $n = n_s + n_u$. Also, in the nonlinear system, there exist stable and unstable manifolds of $\bar{x}$, $W^s(\bar{x})$ and $W^u(\bar{x})$ which are smooth and tangent to E^s and E^u at $\bar{x}$. These manifolds $W^s(\bar{x})$, $W^u(\bar{x})$ are invariant in the sense that a trajectory initially in this manifold stays within it. They are composed of unions of solution curves and provide nonlinear analogues of the flat stable and unstable eigenspaces E^s, E^u of the linearized system.

For the fixed value of time $t = T$, the nonlinear system and its flow give rise to a nonlinear map

$$x_{n+1} = F(x_n) \tag{2}$$

where $F = \phi_T$ is a nonlinear vector valued function. It must be noted that, while the orbit or trajectory $\phi_t(p)$ of a flow is a curve in R^n, the orbit $\{F^n(p)\}$ of a map is a sequence of points. This distinction is shown in Figure 1, where $F^2(p)$ means $F[F(p)]$ and, similaryly, $F^n(p)$ means the nth iteration of the map of p.

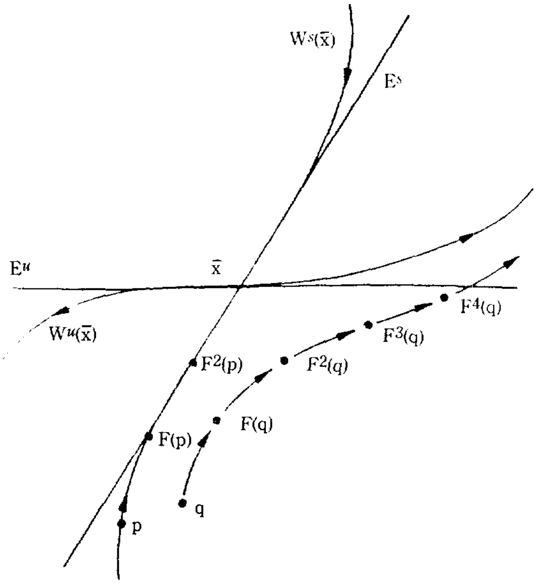

Fig. 1. Invariant manifolds, eigenvectors and maps for a flow.

The stability of the fixed point is also determined by the eigenvalues of the linearized map of F. Let DF $(\bar{x})$ be the $n \times n$ Jacobian matrix of first partial derivatives of the function F at $\bar{x}$. If $\bar{x}$ is a fixed point of $F[F(x) = \bar{x}]$ and $DF(\bar{x})$ has no eigenvalues of unit modulus, $\bar{x}$ is called hyperbolic. If all the eigenvalues have moduli <1, $\bar{x}$ is stable and called a sink or attractor. If any of the eigenvalues has modulus >1 and others have moduli <1, $\bar{x}$ is an unstable saddle. If all the eigenvalues have moduli >1, $\bar{x}$ is a source. The numbers of eigenvalues of which moduli are less than or greater than 1 represent respectively the dimensions of stable and unstable manifolds of $\bar{x}$. However, one must bear in mind that the linearized map or system can only characterize the local structure of a system.

2. Closed Orbits and Poincaré Maps

The dynamical features may appear quite differently depending on the structure of invariant manifolds of the steady states. Here, by a "steady state", we mean one that remains time invariant on the average, and can thus be stationary or periodic depending on whether it is represented by a point or closed orbit. Qualitative changes in the dynamic features occur when a parameter crosses a critical boundary of domains in parametric space. This qualitative structural change in

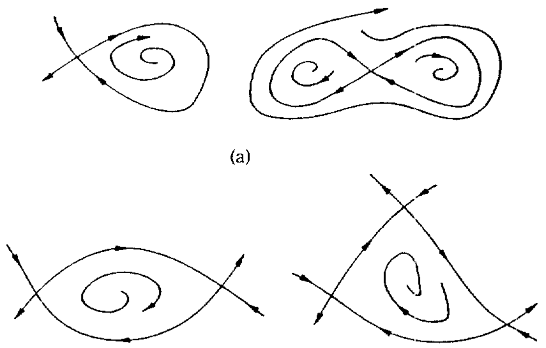

Fig. 2. Some of limit sets for flows. (a) homoclinic orbits, (b) Heteroclinic orbits.

steady state is called "bifurcation". Since the complex dynamical features are closely related to the cyclic time evolution of a system, we may refer to some bifurcation theorems concerning closed orbits.

The Hopf bifurcation (from stationary to periodic) occurs when a spiral sink loses its stability with continuously changing parameters[26]. The Hopf transition can be obtained by seeing when the Jacobian matrix of the linearized system has a pair of complex eigenvalues crossing the imaginary axis. The dynamical behavior, however, is characterized by an invariant manifold tangent to the center eigenspace, called the center manifold[26,27]. In this bifurcation, a limit cycle surrounding an equilibrium point typically emerges from the equilibrium.

Another bifurcation type is associated with a homoclinic orbit for which the unstable manifold of a hyperbolic saddle point returns to itself transversely with infinite period (Fig. 2). When a parameter crosses a boundary through such a point there exists a family of periodic orbits[28,29]. In two-dimensional systems a homoclinic bifurcation can only involve simple limit cycles. However, when the system is three-dimensional or higher, the stable and unstable manifolds may appear tortuously tangled each other and thus the dynamic feature becomes very complicated. Silikov [30,31] suggested the cases when the chaotic motions can be found around the homoclinic orbits in three or higher dimensional systems (Fig. 3). More complex dynamics of a hyperbolic saddle point is found in Lorenz equation[3,32] for which the unstable manifold lies in its stable manifold and we consequently have two symmetric homoclinic orbits circulating over the branches as shown in Figure 4. Thus the appearance of a homoclinic orbit in systems of high dimensions requires our attention in finding chaotic motions. However that does not imply the general existence of

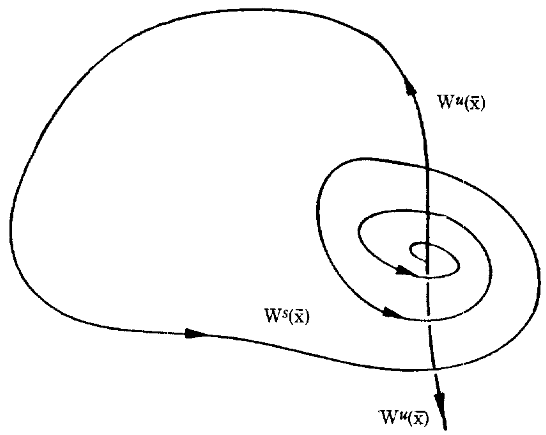

Fig. 3. Homoclinic orbit with two-dimensional stable manifold spiraling to the saddle point.

chaos since the global properties of the flow play a fundamental role in the dynamics.

The dynamical structures of periodic or oscillatory motions are frequently discussed in terms of Poincare map (or first return map), which is important in understanding the geometrical view of phase flows. We first take a local cross section S in n-dimensional real space R^n transverse to a flow ϕ_t and consider point x in S. Then the Poincaré map is defined as the intersection of $\phi_t(x)$ with S(Figure 5). Furthermore, if we let x_o be a point on a periodic orbit with period T, then there exists a unique real valued function $t=\tau(x)$ such that $\phi_t(x) \in S$ with $T=\tau(x_o)$, and x is in a small neighborhood of x_o in S.

This theorem can be used to locate a closed orbit by regarding τ as a continuously differentiable function in the neighborhood of x_o[33,34]. The stability of closed orbits is also determined by the characteristic or Floquet multipliers, which are the eigenvalues of the Jacobian matrix of the Poincare map in the periodic

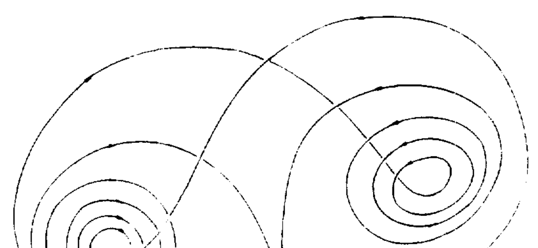

Fig. 4. Lorenz attractor with double loops of homoclinic orbits.

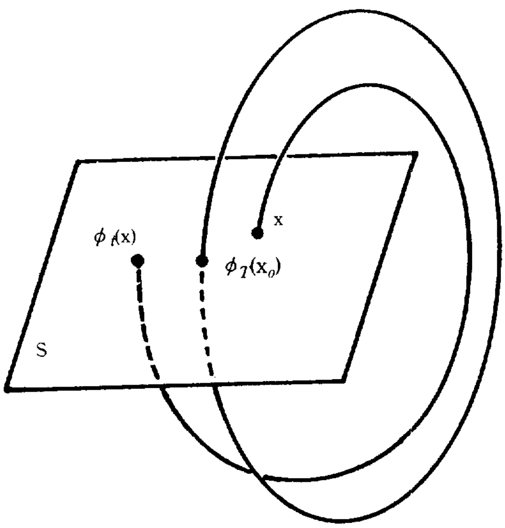

Fig. 5. Poincare maps.

orbit. One must note that one of the multipliers is always on the unit circle at +1.

3. Bifurcation Aspects of Periodic Solutions

Chaotic motion is a type of time evolution of a dynamical system with seemingly stochastic character of self-oscillation, on which all the paths are unstable and behave in a complicated and tortuous fashion. In many cases this change proceeds by succession of period doublings of the periodic motion to some limit, beyond which the attractor changes character and becomes chaotic. Further change in the parameter can lead to an inverse process, sudden disappearance of periodic motion or the appearance of periodic state with K oscillations per period for all natural numbers K. Each of these K-cycles may undergo its own period doubling sequences. This type of universal sequence occurs typically beyond the accumulation point of the 2^n-sequence[35,36].

The period doubling or flip bifurcation occurs when any of the Floquet multipliers leaves the unit circle through -1. Then the periodic solution becomes unstable and the unstable manifold forms a Mobius band such as can be obtained from an open strip by twisting half turn and connecting both ends. The trajectories on the surface of the band tend to the boundaries and form a stable cycle with the period almost twice the original periodic orbit (Fig. 6).

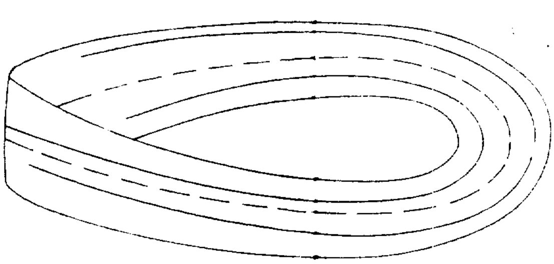

Fig. 6. Period doubled limit cycle on the Mobius band.

This period doubling bifurcation usually occurs repeatedly and it is notable that substantial evidence was developed empirically and theoretically for a kind of universal constant which says that the period doubling bifurcations occur on a shrinking scale in parameter μ such that the ratio

$$\delta_n = (\mu_n - \mu_{n-1})/(\mu_{n+1} - \mu_n) \tag{3}$$

approaches to a constant value of 4.669201... as n goes to infinity. It is called Feigenbaum series[37,38] and known to be independent of the nature of the system and holds for most nonlinear transformations.

This sequence of period doubling bifurcations can be observed in one-dimensional noninvertible map as was used by Feigenbaum. We may consider the function $F(x_n)$ in Equation (2) as

$$F(x_n) = \alpha x_n (1 - x_n), \quad 0 \leqq x_n \leqq 1 \tag{4}$$

where α is a parameter. This may be considered as a sequence of Poincaré maps of an intergrated ordinary differential equation, $dx/dt = f(x)$. The fixed point is defined as the solution of

$$\bar{x} = F(\bar{x}) \tag{5}$$

and the stability of the fixed point $\bar{x}$ is determined by the Jacobian matrix of the map at $\bar{x}$, $F'(\bar{x})$ as we have discussed in previous section. Therefore, the stability region in parameter space should satisfy the condition

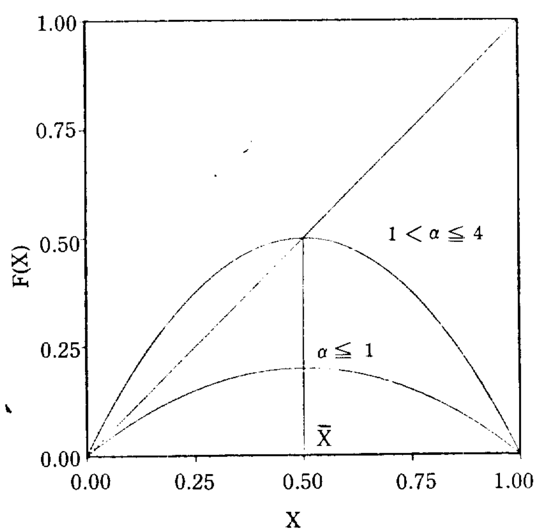

Fig. 7. One-dimensional iteration map $F(x_n)$.

$$|F'(\bar{x})|<1 \quad (6)$$

Thus if $|F'(\bar{x})|>1$, the maps F of points near $\bar{x}$ successively move farther away from it, and $\bar{x}$ is unstable.

Now we consider α to increase from zero. When α is less than 1, we have only one fixed point at $x=0$, and the zero is stable. When α increases beyond 1, the zero point becomes unstable and another stable fixed point appears at $\bar{x}=1-1/\alpha$(Fig. 7). The slope $F'(\bar{x})$ decreases from 1 as α increases, and then crosses -1 at $\alpha=3.0$. In this case two branches of points appear recursively every second iteration. This can be expressed with period two maps,

$$x_{n+2}=F^2(x_n) \quad (7)$$

Figure 8 illustrates the period one and two maps at $\alpha=3.1$ with two fixed points ($\bar{x}=0.5580$, 0.0746). The stability of the fixed points is then determined by the condition,

$$|F^{2\prime}(\bar{x})|<1 \quad (8)$$

Two branches of period two points are stable until $\alpha=3.44948$ and then, as can be deduced, two branches of period four points appear for each of period two branches. Figure 9 shows the four fixed points ($\bar{x}=0.3828$, 0.8269, 0.5009, 0.8750) at $\alpha=3.5$ for period one and period four maps.

In this way sequential bifurcation of period doublings propagates until $\alpha=3.57$ with $\delta_\infty=4.6692$ as Feigenbaum has derived. After the chaotic regime, odd period points begin to appear at $\alpha=3.6786$ and at $\alpha=3.8284$ there appears a period three point, at which point there can be points of any period[39]. These odd number of periodic points undergo their own period doubling bifurcations. More precise explanation may be referred to the literatures[40,41].

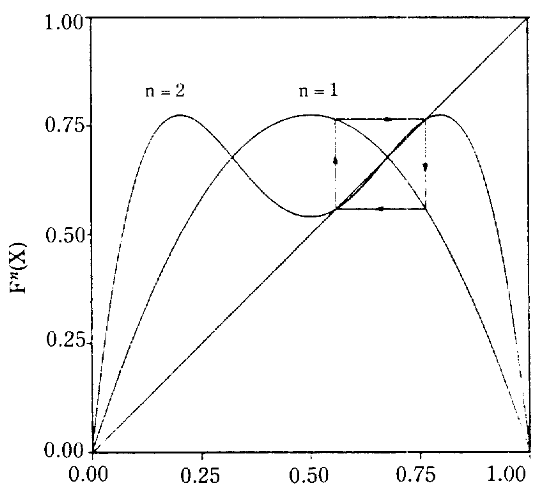

Fig. 8. Stable period two cycle and F^2 map for $\alpha=3.1$.

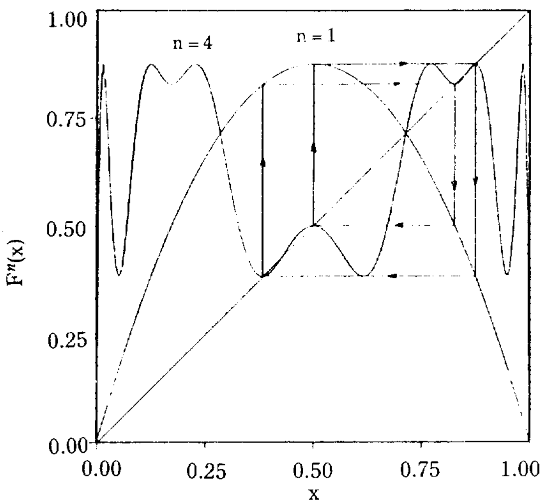

Fig. 9. Stable period four cycle and F^4 map for $\alpha=3.5$.

STRANGE ATTRACTORS

1. Typical Properties

When the chaotic motion is discussed, the question may arise as to how this strange behavior can appear and what the nature of that motion is. The answer, though it may have some distance from complete solution, lies in the instability of the solutions for the system. For better understanding of the stochastic nature, physical and topological insight may be emphasized rather than mathematical rigor. We are now going to discuss some characteristic features of complex dynamics.

Referring to the phase portrait of the chaotic mo-

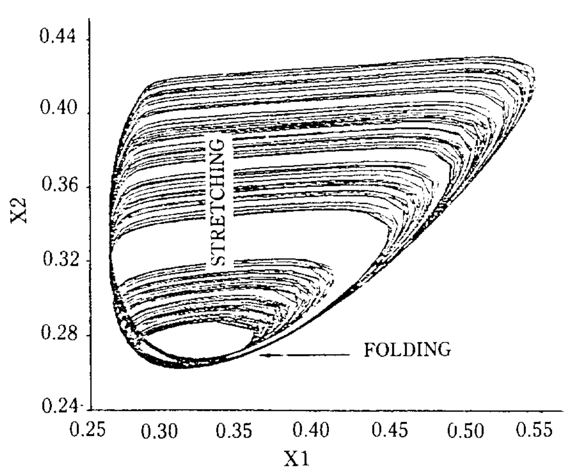

Fig. 10. Phase flow of a strange attractor on a plane.

tion on a strange attractor as shown in Figure 10[36], one can visualize the flow in the form of layers of two dimensional sheets and the layer stretches along its width and folds over on itself. This property is called the hyperbolicity of strange attractor and is related to other typical properties of chaotic motion such as Smale horseshoe, Cantor property and the divergency of trajectories. The appearance of these features in a dynamical system allows us to assume that the complex modulation regimes correspond to a strange attractor in the phase space.

The horseshoe is such that a rectangle in the intersecting plane is mapped into a curved figure reminiscent of a horseshoe. When we consider a Poincaré map from the square segment S, $F: S \to R^2$, the Jacobian of F can be thought of as performing vertical expansion and horizontal contraction of S by the factors of the eigenvalues of DF (Fig. 11). The reiteration of mapping leads the images on S into finer and finer scales of a leaflike pattern. The formation of the typical leaf structure is characterized mathematically as a Cantor set. The leaved structure across the layer can be clearly seen in an example due to Hénon[42], who considered the case of a two-dimensional quadratic mapping,

$$x_{n+1} = y_n - a x_n + 1$$
$$y_{n+1} = b x_n \tag{9}$$

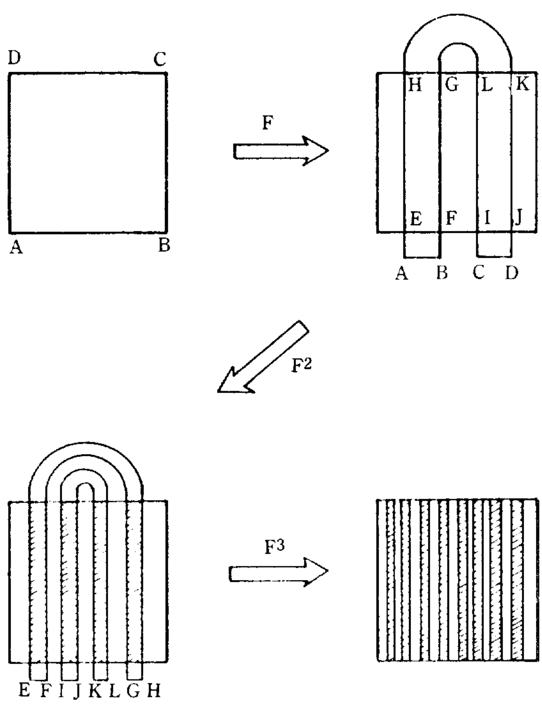

Fig. 11. The iterated maps of a square which reminisces the horseshoes.

This invertible mapping may be thought of as the Poincaré section of a three dimensional flow system. When $a = 1.4$ and $b = 0.3$, the map shows the finer structure of leaves (Fig. 12) and the area contracts by the factor $|\det DF|$ for each intersection.

The divergency property in complex attractors is that the nearby trajectories diverge farther and farther away from each other with successive maps, implying that all the trajectories on the attractor are unstable. The divergency, in a statistical sense, may be thought of as an invariant measure describing quantitative properties of dynamical systems. In dealing with this problem we will examine why the measure relies on the mathematical rigor and how the numerical computations describe the real system behavior.

2. Invariant Measure

To fully understand the concepts of invariant measure, one may need the background for the ergodicity [43], which gives us probabilistic information to describe quantitative properties of dynamical systems. If we let $F: R^n \to R^n$ be a discrete dynamical system and let $g: R^n \to R$ be a real valued function, the time average of g on the trajectory of x is defined as

$$\bar{g}(x) = \lim_{N\to\infty} (1/N) \sum_{i=0}^{N-1} g(F^i(x)) \tag{10}$$

and is invariant for all initial x. The ergodicity of dynamical systems is defined if

$$\bar{g}(x) = \int g(x) P(x) \, dx \tag{11}$$

where P(x) is the probability or invariant distribution.

For a system corresponding to an n-periodic motion, P(x) appears discrete, consisting of n δ-functions at the n stable fixed points of the map. When the motion is chaotic, P(x) appears nonzero over a finite range of x even though it may be discontinuous. The probability distribution P(x) can be constructed numerically from the equation

$$P(x) = P(x_1)/|dF/dx|_{x_1} + P(x_2)/|dF/dx|_{x_2} \tag{12}$$

where x_1 and x_2 are the points of inverse mapping for x. Figure 13[44] shows the iterated maps and P(x) for one dimensional map of Equation(4) with $a = 3.825$, displaying the chaotic behavior for a finite range of x. In a similar sense, we may consider the topological entropy which is stochastic indicator of a dynamical system. Let us define $\varepsilon > 0$ and an integer $n > 0$, and let $M(\varepsilon, n)$ be the maximum number of different paths separated by a distance greater than ε, i.e., for two different paths x_1, and x_2, there exists $0 < i < n$ such that $d[F^i(x_1), F^i(x_2)] > \varepsilon$. Then, the topological entropy of the dynamic system is given gy

$$h(F) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \ln M(\varepsilon, n)/n \tag{13}$$

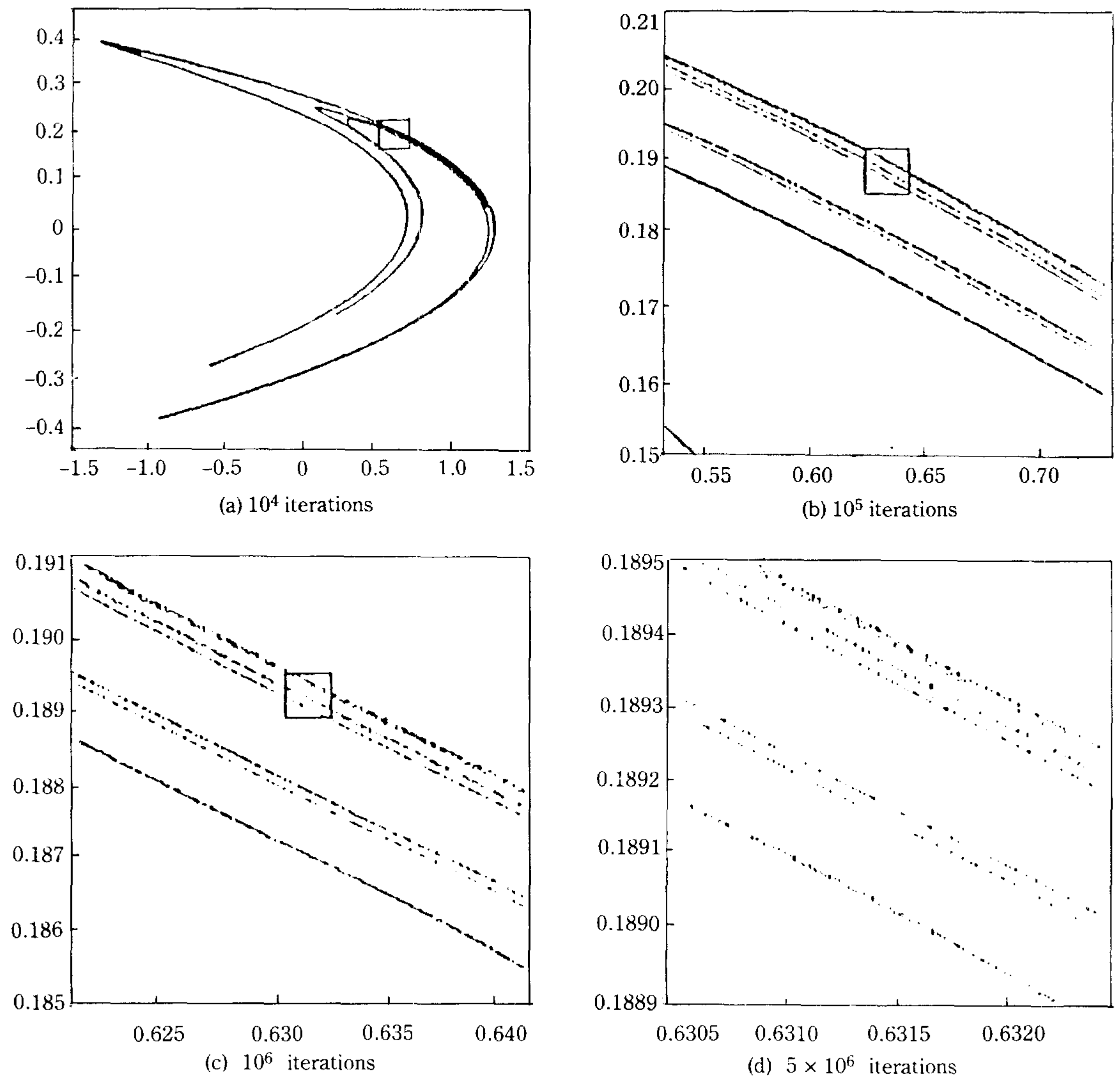

Fig. 12. Hénon attractor.

From this definition, it immediately follows, in particular, that if a flow path is stable in the sense of Liapunov, the entropy is zero and $M(\varepsilon,n)$ does not increase with increasing n. If h is greater than zero for the system, it is natural to refer to the dynamic behavior as stochastic. We my note here that the topologic entropy is an invariant measure of stochasticity, which means two dynamical systems having the same metric entropy are related by an isomorphism that preserves measure.

When we replace g(x) of Equation(11) with $\ln|dF/dx|$ we obtain the largest Liapunov exponent which allows one to define a quantitative parameter like an entropy. The Liapunov characteristic exponents measure the average asymptotic divergence rate of nearby trajectories in different directions of a system's phase space and will be discussed further in some detail.

3. Liapunov Exponents

Let us define the m-dimensional system as before

$$dx/dt = f(x),\ x(0) = x_o \tag{14}$$

Consider a trajectory in m-dimensional phase space and a nearby trajectory with initial condition x_o and $y_o = x_o + \Delta x_o$ respectively. These evolve with time τ yielding the tangent vector $\Delta x(x_o,\tau)$ with its Euclidean norm $d(x_o,t) = \| \Delta x(x_o,\tau)\|$,

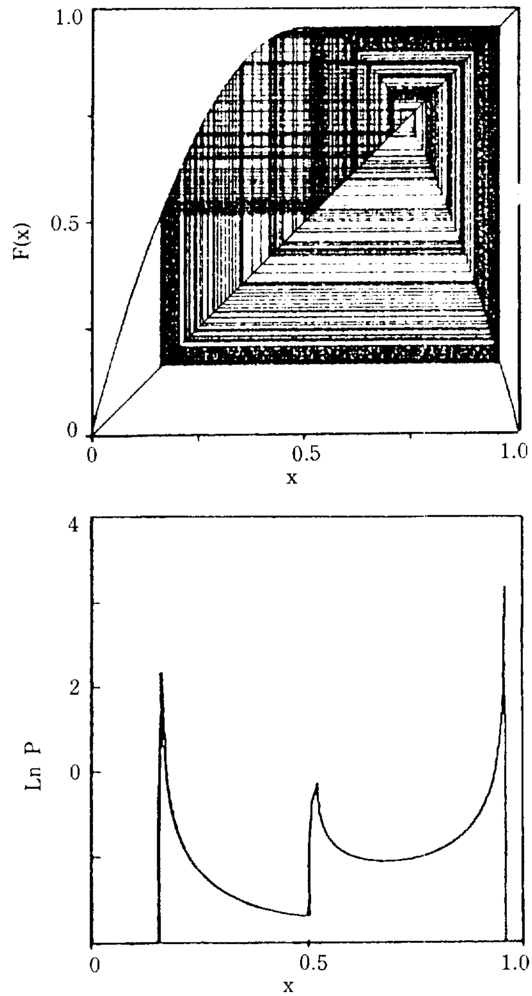

Fig. 13. One-dimensional map and invariant distribution P(x) for $\alpha = 3.825$.

$$\Delta x = \int_0^{\tau} [f(y) - f(x)]\, dt + \Delta x_o \tag{15}$$

Writing for convenience $u = \Delta x$, the time evolution for u can be expressed approximately by linearizing (15) along the trajectory $x(x_o,t)$ with the assumption that u is small

$$du/dt = J[x(t)] \quad u \tag{16}$$

where $J = \partial f/\partial x$ is the Jacobian matrix of f. Then the mean exponential divergence rate of initially close trajectories is

$$\sigma(x_o, u_o) = \lim_{t\to\infty} 1/t \ \ln[d(x_o, t)/d(x_o, 0)] \tag{17}$$

Furthermore, there is an m-dimensional basis $\{\hat{e}_i\}$ of u such that for any u, σ takes on one of m values $\sigma_i(x_o) = \sigma(x_o,\hat{e}_i)$, which are the Liapunov characteristic exponents and can be ordered by size.

When the Equation (16) and (17) are used, the Liapunov exponents can be obtained. However, in chaotic motion, the norm u increases exponentially with increasing time, and this leads to the problem of overflow and other computation errors. To circumvent this problem, one can use the renormalization of u to the unity after every finite time [46,47], and obtain the mean value of σ_n as

$$\sigma_n = 1/(n\tau) \sum_{i=1}^{n} \ln u_i(\tau) \tag{18}$$

Liapunov exponents are also defined for maps as well as flows. Let us consider the m-dimensional map

$$x_{n+1} = F(x_n) \tag{19}$$

and let us introduce the eigenvalues $\lambda_i(n)$ of the matrix

$$A_n = [J(x_n) \cdot J(x_{n-1}) \cdots J(x_1)]^{1/n} \tag{20}$$

where J is the Jacobian matrix of the map, $\partial F/\partial x$. Then the Liapunov exponents are given by

$$\sigma_i = \lim_{n\to\infty} \ln|\lambda_i(n)|, \quad i = 1, 2, \cdots, m \tag{21}$$

Therefore, the Liapunov exponents for a flow can be obtained on the Poincaré section weighted by the mean time of successive iterations.

Since the chaotic nature of dynamics is revealed by the divergence of the nearby trajectories, the largest Liapunov exponent always shows positive values like the topologic entropy. While for a periodic orbit, the largest exponent will obviously die away when the trajectory returns to the same point at every period. When the largest exponent converges to a negative value, we refer to it as the case of non-periodic attractor.

In computing the Liapunov exponents for flows, however, there exist problems concerning the fractal nature of chaotic attractor, which prohibits smooth mapping from the flows and brings about noise in calculation. Referring to the dimensions of an attractor which is also a clear measure to characterize its property, a strange attractor typically bears the dimension slightly greater than 1[45]. For this reason, even though one-dimensional map is constructed from a flow, the Liapunov exponents are very sensitive to the noise [48], and thus may rely upon the method of numerical computation.

CONCLUSION

We have briefly described the stochastic motions of nonlinear oscillatory systems. From this article one obviously cannot be expected to obtain a full understanding of this rapidly growing field of stochastic systems and their applications in a number of areas in science and engineering. However the authours hope to have made some of the fundamental concepts of complex dynamics and the ways of application to the engineering problems.

This subject has received most attention from physical scientists and mathematicians. But now the existence of strange attractors in the phase space of nonlinear systems has turned out to be almost as common as the existence of limit cycles. Furthermore, it has been known that this exotic motion happens because of the instability and the tangling of paths within the attractor. This implies that perturbation of state however small will never produce the same trajectory (divergence), and yet the qualitative features are maintained (structural stability). However it is still not easy to locate the stochastic motions exactly even with the clue of Feigenbaum sequence.

Turbulence in fluid flows represents the stochastic regimes of self-oscillations in partial differential systems. The full solution of this problem has not yet been obtained, and the Lorenz system derived from two partical differential equations[3] does not describe the problem exactly even though it has made a remarkable contribution to the study of this field. For chemically reacting systems, oscillatory motions are to be found in distributed systems[49-51]. However, it is not clear that they undergo sequential period doubling bifurcations and culminates in stochastic motions at the accumulation point. These remain as problems yet to be solved for research in this area is still just beginning.

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