TURBULENCE OF A NON-NEWTONIAN FLUID

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Abstract—This paper presents a simple theory for a non-Newtonian fluid, especially the corotational Jeffreys model. Particular attention is paid to the frequency spectrum of the strain fluctuations, and through this article it is found that the Jeffreys fluid will exhibit an "onset" Reynolds number, above which the effects of the non-Newtonian nature of this fluid are felt. Because time dependent behavior of the strain-strain correlation is emphasized, this study is complementary to the molecular theory.

INTRODUCTION

There has been a great resurgence of interest in the turbulence of non-Newtonian fluids. This is probably not only because many turbulent flow involve non-Newtonian liquids, such as slurries and vapor-liquid mixtures, but also because there exists a vast potential for energy savings through the exploitation of polymer-induced turbulent drag reduction [1]. It would be useful, therefore, to produce a simple analogy between a turbulence involving the Newtonian constitutive relation and one involving a physically realistic non-Newtonian relation.

Much of what is understood about Newtonian turbulence has been gained by careful study of the kinetic energy (KE) budget of the flow. This budget concerns itself primarily with accounting for processes that convey energy to the turbulence and processes by which that energy is transmitted and dissipated. For the KE budget of a normal Newtonian fluid, terms arising from the Navier-Stokes equation can be identified for each of these processes[2]. It would seem, then, that scrutinizing this KE budget might also be profitable with regard to a non-Newtonian turbulence.

In what follows we will consider a corotational model of the non-Newtonian liquid[3]. We will be able to exploit the "finite energy of the eddies" maxim, and to some extent our own ignorance about the detailed statistics of the flow, to come up with a KE budget analogy for this fluid. Since any constitutive relation is unlikely to affect the inertial properties of the fluid, it is not suprising that we find the major revision due to the

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presence of the Jeffreys fluid in the dissipation portion of this budget. As a result, we will be able to make some conjectures about how the statistics of the turbulence may be affected by the "memory" involving characteristics of this fluid. None of these inferences are possible for the more realistic codeformational models however, and, in the conclusion, we will provide some speculation on the differences between the corotational (e.g. Jeffreys) and codeformational models.

KINETIC ENERGY (KE) BUDGET EQUATION

1. Newtonian fluid

We will start by deriving the turbulent energy equations for a Newtonian fluid and then quickly derive its analogy for a non-Newtonian fluid. The equation for the mean energy in turbulence of Newtonian fluid is[4]:

$$\frac{\partial}{\partial t}(1/2 U_i U_i) =$$

$$U_j \frac{\partial}{\partial x_j} (\frac{-P}{\rho} U_i + \frac{2\mu}{\rho} U_i S_{ij} - U_i <u_i u_i>) +$$

$$\frac{-2\mu}{\rho} S_{ij} S_{ij} + S_{ij} <u_i u_i>, \qquad (1)$$

where it is noted that repeated indices imply summation (Einstein convention). This is obtained by taking the scalar product U_j (the mean velocity) into the averaged equation of motion with the underlined terms being a direct result of the Newtonian constitutive assumption. Here, S_{ij} represents the mean strain rate and P the average pressure. Lower case symbols indicate the fluctuating components: u_j is the fluctuating velocity ($<u_j>= 0$) and μ is the viscosity of fluid. For

future reference, we also define p as the fluctuating pressure ($<p> = 0$), and s_{ij} as the fluctuating strain ($<s_{ij}> = 0$).

Now, if we take the scalar product ($U_j + u_j$) into the (unaveraged) equation of motion and perform the averaging operation:

$$\frac{\partial}{\partial t}(1/2 U_i U_i + 1/2 <u_i u_i>) + 1/2 U_j \frac{\partial}{\partial x_j}(U_i U_j$$
$$+ <u_i u_i>) + \frac{\partial}{\partial x_j}(<u_j u_i u_i> + \underline{2 <u_j u_i> U_i})$$
$$= \frac{-1}{\rho} U_i \frac{\partial P}{\partial x_i} + \frac{-1 \partial}{\rho \partial x_i} <u_i p> + \frac{2\mu}{\rho} U_i \frac{\partial S_{ki}}{\partial x_k}$$
$$+ \underline{\frac{2\mu}{\rho} <u_i \frac{\partial s_{ki}}{\partial x_k}>}. \tag{2}$$

By subtracting off the mean energy [eq. (1)], we obtain the turbulent energy equation for the eddies:

$$\frac{\partial}{\partial t}(1/2 <u_i u_i>) + U_j \frac{\partial}{\partial x_j}(1/2 <u_i u_i>) =$$
$$- \frac{\partial}{\partial x_k}(1/2 <u_k u_i u_i> + \frac{<u_k p>}{\rho} - \frac{2\mu}{\rho} <u_i s_{ki}>)$$
$$- <u_i u_k> S_{ki} - \frac{2\mu}{\rho} <s_{ki} s_{ki}>. \tag{3}$$

The last two terms are the most important, because they are the "source" and "sink" terms. See Hinze[2] for the detailed meaning of these terms.

2. Non-Newtonian fluid

What we wish to do now is to generalize the derivation for the Newtonian fluid with an arbitrary constitutive relationship. Let the total stress be given by:

Total stress = Isotropic pressure + Deviatoric stress

$$= - (P + p) \underset{\approx}{\delta} + \underset{\approx}{T} + \underset{\approx}{\tau}. \tag{4}$$

Here, P and $\underset{\approx}{T}$ are the average pressure and deviatoric stress, whereas p and $\underset{\approx}{\tau}$ are the fluctuating pressure and deviatoric stress, and $\underset{\approx}{\delta}$ is a unit tensor. The deviatoric stress ($\underset{\approx}{T} + \underset{\approx}{\tau}$) is functionally related to the rate of strain tensor ($\underset{\approx}{S} + \underset{\approx}{s}$). This constitutive relationship is known as "rheological equation of states" (RES) and is available in standard texts (see, for example, Bird et al.)[3].

In this section, we will derive a general formulation suitable for any RES. Here, we shall draw the analogy between the non-Newtonian fluid and the Newtonian fluid to avoid any redundancy in our calculations. Then, the Newtonian viscous stress term in the equation of motion is replaced with $\frac{1 \partial}{\rho \partial x_i}(T_{ki} + \tau_{ki})$. In the mean energy equation, the underlined terms in eq.(1) are replaced by

$$\frac{1 \partial}{\rho \partial x_i}(U_k T_{ki}) - \frac{1}{\rho} S_{ki} T_{ki}.$$

The underlined terms in eq.(2) are then replaced by

$$\frac{1 \partial}{\rho \partial x_i}(U_k T_{ki} + <u_k \tau_{ki}>) - \frac{1}{\rho}(S_{ki} T_{ki} + <s_{ki} \tau_{ki}>).$$

After subtracting the new mean energy equation, we obtain:

$$\frac{\partial}{\partial t}(1/2 <u_i u_i>) + U_j \frac{\partial}{\partial x_j}(1/2 <u_i u_i>) =$$
$$- \frac{\partial}{\partial x_j}(1/2 <u_j u_i u_i> + <u_j p/\rho> - <u_k \tau_{kj}>)$$
$$- 1/\rho <\tau_{ki} s_{ki}> - <u_i u_j> S_{ij}. \tag{5}$$

Eq.(5) is the turbulent energy equation for the eddies, and it is applicable for any RES. For the Newtonian fluid, $\underset{\approx}{\tau} = 2\mu \underset{\approx}{s}$, eq.(5) reduces to eq.(3) as expected. The dissipation term, $D \equiv (\tau_{ki} s_{ki})$ plays an important role in the study of turbulence, and will be studied in conjunction with the Jeffreys model, as an example, and it will be shown in the next section.

STUDY OF ENERGY DISSIPATION FOR JEFFREYS FLUID

The Jeffreys model is, possibly one of the simplest constitutive equations which involve the "memory" or the history of the strain field and which still preserve objectivity*. The defining equation can be written formally so as to simplify the analysis; we assume ($\underset{\approx}{T} = \underset{\approx}{S} = 0$):

$$\tau + \lambda_1 \frac{\mathscr{D}\tau}{\mathscr{D}t} = 2\mu (s + \lambda_2 \frac{\mathscr{D}s}{\mathscr{D}t}) \tag{6}$$

where, $\mathscr{D}/\mathscr{D}t$ is the corotational derivative and,

$$\frac{\mathscr{D}A}{\mathscr{D}t} = \frac{DA}{Dt} + \underset{\approx}{\omega} \cdot A - A \cdot \underset{\approx}{\omega}.$$

Here, $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ is the Stokes derivative, and $\underset{\approx}{\omega} = 1/2(\nabla \mathbf{v} - [\nabla \mathbf{v}]^T)$ is known as the vorticity tensor.

We can rewrite this equation by the use of an orthogonal transformation that rotates with the fluid:

$$\tau' + \lambda_1 \frac{D\tau'}{Dt} = 2\mu (s' + \lambda_2 \frac{Ds'}{Dt}),$$

where $\mathbf{A}' = \mathbf{Q}\mathbf{A}\mathbf{Q}^+$ and $\dot{\mathbf{Q}} = \underset{\approx}{\omega}\mathbf{Q}$ with $\mathbf{Q}(0) = \mathbf{1}$, or, formally

$$\mathbf{Q}(t) = \exp_+ \{ \int_s^t d\xi [-\underset{\approx}{\omega}(\xi)] \}.$$

Here, the "+" subscript indicates time ordering from left to right. In this equation, we consider $\underset{\approx}{\omega}$ to be convected with the fluid point. It now becomes important to identify what is meant by convected coordinates and to define the notation we will be using.

Define $\mathbf{x}$ to be the coordinate of a fluid particle at some previous time θ, and $\mathbf{q}$ to be the coordinate at

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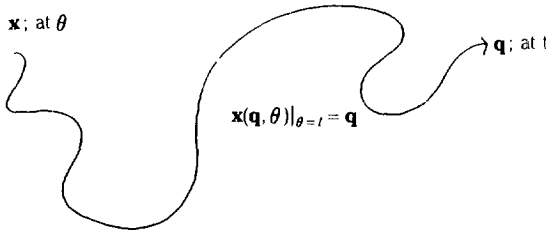*Called the Principle of Material Frame Indifference (see ref. 5 and 6).

Fig. 1.

the present time t (see Figure 1):

More precisely, we can say that $x(q, \theta)$ is the trajectory of the particle. This puts us in a position to define **Q** unambiguously:

$$Q(q, t) = \exp_+ \{ \int_\theta^t d\xi \, [-\underline{\underline{\omega}}(x(q, \xi), \xi)] \}. \tag{8}$$

Now, taking the Fourier transform of eq. (7), we can solve for τ' in Fourier space:

$$\bar{\tau}' = 2\mu \frac{1 - \lambda_2 i \omega}{1 - \lambda_1 i \omega} \bar{s}'. \tag{9}$$

Here barred variables denote Fourier transformed ones.

Even for a Newtonian turbulence, the Fourier picture has shown great utility in the study of time dependent statistics, and the Fourier representation here is particularly simple. In any case, the real space τ can be represented by a convolution;

$$\tau = \int_{-\infty}^{\infty} d\theta G(t - \theta) s'(\theta), \tag{10}$$

where $\bar{G}$ is the Fourier transform of G, and:

$$\bar{G} = 2\nu \frac{1 - \lambda_2 i \omega}{1 - \lambda_1 i \omega}. \tag{11}$$

Here, $\nu \equiv \mu / \rho$ is the kinematic viscosity.

The dissipation at some point in the fluid **q** is defined as:

$$D \equiv <\tau_{kl} s_{kl}> = <\tau_{kl}(q, t) s_{kl}(q, t)>. \tag{12}$$

We can see from the particular model chosen (the corotational Jeffreys model) that the dissipation at a particular point will depend on its convected strain history. From eq. (10), we can rearrange the (unaveraged) dissipation term into

$$D = \int_{-\infty}^{\infty} d\theta G(t - \theta) s_{ik}'[x(q, \theta), \theta] s_{ik}[x(q, t), t], \tag{13}$$

and, taking the average,

$$<D> = \int_{-\infty}^{\infty} d\theta G(t - \theta) <s_{ik}'[x(q, \theta), \theta] s_{ik}$$
$$[x(q, t), t]>. \tag{14}$$

We will assume that the flow is stationary in t and in **q**, so that $<D>$ is independent of **q**, and we define:

$$\Sigma(t - \theta) = <s_{ik}'[x(q, \theta), \theta] s_{ik}[x(q, t), t]>. \tag{15}$$

From eqs. (14) and (15),

$$<D> = \int_{-\infty}^{\infty} dt G(t) \Sigma(t), \tag{16}$$

and from Parseval's theorem:

$$<D> = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \bar{G}^*(\omega) \bar{\Sigma}(\omega). \tag{17}$$

Here, asterisk denotes the complex conjugate.

Now it is clear that, since $\bar{\Sigma}$ is an even function of ω, we need consider only the even part of $\bar{G}^*$ (Note that the imaginary part of $\bar{G}^*$ is an odd function of ω):

$$\text{Real part of } \bar{G}^*(\omega) = 2\nu \frac{1 + \alpha (\lambda_1 \omega)^2}{1 + (\lambda_1 \omega)^2} \tag{18}$$

where $\alpha = \lambda_2 / \lambda_1$, and $1/3 < \alpha < 1$.

A log-log plot of $\bar{G}^*$ and $\bar{\Sigma}$ might look like the schematic drawing that appears in Figure 2. We not that $\bar{G}^*$ has a relatively sharp step at $\omega = 1/\lambda_1$. If $\bar{\Sigma}$ dies off before $1/\lambda_1$, then it seems that the dissipation will be unchanged from the Newtonian case. It may be assumed that, as the Reynolds number increases, the $\bar{\Sigma}$ curve spreads out and will interact with $\bar{G}^*$ to yield a non-Newtonian behavior. We cannot say as yet what the effect on the $\bar{\Sigma}$ spectrum will be, because the spectrum must rearrange itself in such a way as to preserve the total amount of dissipation. All we can say is that the continuum view presented here exhibits a property similar to the onset characteristic of the molecular analysis done in a previous paper[7].

A similar analysis can be done with the Oldroyd two-constant model[8], using codeformational deriva-
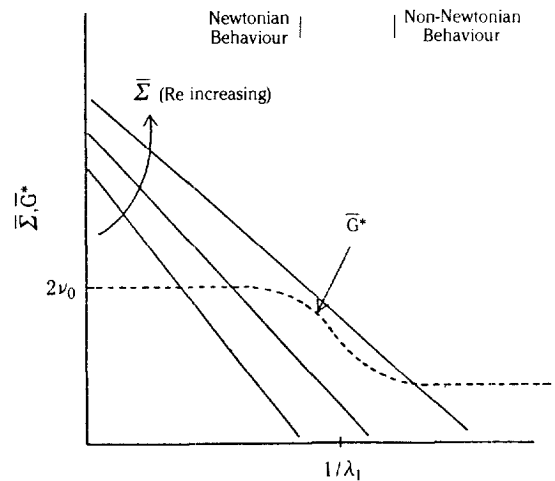

Fig. 2. A plot of the strain correlation $\bar{\Sigma}$ and the Green's function $\bar{G}^*$ versus the frequency ω.
$\bar{\Sigma}$ is the Fourier transform of Σ.

tives instead of corotational ones as done above. However, this would involve a nonorthogonal transformation instead of the orthogonal corotating one used in this section, and would greatly complicate the mathematics. However, the integral formulation of this model is wholly unchanged from the molecular analysis done previously[9]. Indeed, the Oldroyd two constant model is in every way the continuum manifestation of the Hooken dumbbell molecular model[3].

DISCUSSION

What we have here is to follow the progress of a Jeffreys fluid particle in its "natural" coordinate system. Since we have little information about the statistics of the strain-strain correlation function resulting from any kind of turbulence and in any kind of coordinate system, we might as well choose this natural one. This is what is meant by exploiting our ignorance. We can be certain that, in any case, Σ has a frequency scale associated with it, over which it will change significantly and be substantially larger than zero (for a Newtonian fluid this scale would be something like the inverse of the Kolmogorov time scale). Moreover, if the frequency scale of the fluid's memory function $(1/\lambda_1)$ is much longer than that of $\overline{\Sigma}$ (low Reynolds numbers) the turbulence suffered by this Jeffreys fluid will be little changed from its Newtonian counterpart. On the other hand, if the reverse is true-that is, if the time scale of the fluid is much shorter than that of $\overline{\Sigma}$ (high Reynolds number)-the turbulence should behave as if it were a Newtonian fluid, but with a lower viscosity: $\nu\alpha$.

Between these two extremes we cannot say anything precise. If the turbulence is isotropic and the strain fluctuations are assumed to be associated almost solely with the dissipation, or if, in other words, the energy production mechanism is indifferent to the non-Newtonian character of the fluid, then the amount of energy that must be dissipated will be unchanged from a corresponding Newtonian turbulence. In this event, $\overline{\Sigma}$ must rearrange itself on Figure 2 so as to produce as much energy dissipation ($<D>$) as is being produced by the Reynolds stresses. How $\overline{\Sigma}$ might accomplish this is open to debate. It should be emphasized that we are considering only an isotropic turbulence in the above-turbulent shear flows might well be expected to manifest different results. It is a recognized experimental fact, however, that to large degree the convection and diffusion of energy from one point to another in a Newtonian turbulent shear flow is negligible[2]. To the extent that this is also true for a Jeffreys fluid, we could say that at every point in the shear (inertial layer) one would expect that the Newto-

nian energy production matches the dissipation ($<D>$). This contrivance has been exploited in the past as an aid to the understanding of non-Newtonian shear turbulence[9,10] (e.g. drag reduction).

It is clear that this corotational model reflects a history of "natural" rotations of the initial coordinate system. The fact that each of these coordinate systems are orthogonal ensures that no singularity will be incurred. Codeformational models do not enjoy this luxury, and for many such models, even under perfectly deterministic flows (e.g. pure elongational), every invariant of the stress tensor will exhibit a singularity at some finite and physically reasonable strain rate. Lumley[11] showed that there is every reason to believe that the same is true for a stochastic deformation rate. This possibility has been supported further by our studies.

To those familiar with the previous work in this field, it may seem that we are in conflict with the well accepted view of the mechanism for drag reduction. Briefly, this view requires, for a variety of reasons, that the functional "viscosity" of the non-Newtonian liquid increases with "increasing" Reynolds number. We, of course, have predicted quite the opposite and, we believe, this is largely due to the type constitutive relation considered here. Corotational models, when viewed on the microscopic scale, do not allow for a great deal of elasticity -- a characteristic particularly important in polymeric solutions. In fact, codeformational models have been shown to increase the amount of energy dissipation (and therefore, by a crude analogy, the functional "viscosity") at high Reynolds numbers[7]. So it seems that liquids which are better by a Jeffreys model may behave in a fundamentally different manner than polymeric systems with a large amount of elasticity.

So why not develop the same treatment given here for a codeformational fluid? Codeformational models (although linear in the stress tensor) are nonlinear in strain, and this single fact renders the problem nonlinear and stochastic, with philosophical as well as mathematical difficulties. In addition, the inherent singularity mentioned earlier may cause such a model to be an unphysical representation of real non-Newtonian fluids, especially if the strain fluctuations encountered due to the turbulence are expected to be large. Therefore, the corotational models are probably the only memory involving constitutive relation that can be applied to this type of continuum mechanical treatment. One must proceed to a nonlinear molecular theory in order to do away with this problem[9,12].

This paper then, has dealt with the most realistic (memory involving) fluid that does not suffer from these drawbacks. The major advantage to this con-

tinuum view over the molecular view is that a "white" frequency spectrum assumption for $\bar{\Sigma}$ (high Reynolds number limit) is not required and the frequency dependent properties of the resulting non-Newtonian turbulence can therefore be studied more realistically. To some extent we have compromised a little of the realism of the constitutive relations in order to obtain a clearer idea of what happens to the turbulence, and in that sense this work is we developed[1,7] somewhat complementary to the previous molecular theories.

NOMENCLATURE

D	:	dissipation
G	:	relaxation modulus
$\bar{G}$	:	Fourier transform of G
p	:	pressure fluctuation
P	:	average pressure
q	:	coordinate at the present time t
$\mathbf{s}(s_{ki})$	:	fluctuating rate of strain tensor
$\mathbf{S}(S_{ki})$	:	average rate of strain tensor
s_{ij}	:	strain rate fluctuation
S_{ij}	:	mean strain rate
$\bar{\mathbf{S}}'$	:	Fourier transform of $\mathbf{S}'$
t	:	present time
$\mathbf{T}(T_{ki})$	:	average deviatoric stress
u_i	:	velocity fluctuation
U_j	:	mean flow
$\mathbf{X}$	:	coordinate of a fluid particle at some previous time θ
Re	:	Reynolds number
$\dfrac{\mathscr{D}}{\mathscr{D}t}$	:	corotational derivative
$\dfrac{D}{Dt}$	:	Stokes derivative
$< >$	:	average quantity

Greek Letters

α	:	λ_2/λ_1
ρ	:	fluid density
λ_1	:	relaxation time
λ_2	:	retardation time
ν	:	kinematic viscosity (μ/ρ)
μ	:	fluid viscosity
θ	:	previous time
$\bar{\Sigma}$	:	Fourier transform of Σ
$\underset{\approx}{\tau}(\tau_{ki})$	:	fluctuating deviatoric stress
$\bar{\tau}'$	:	Fourier transform of τ'
ω	:	frequency
$\underset{\approx}{\omega}$	:	vorticity tensor

Superscripts

+	:	time ordering from left to right
*	:	complex conjugate

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