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NOTE

STUDY ON THE LUBRICATION APPROXIMATION FOR POWER LAW FLUIDS

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Abstract—To appreciate the effectiveness of lubrication approximation for the non-Newtonian fluid, power law fluid flowing between nonparallel plates was investigated under condition with no inertia. First the flow problem was successfully reduced to a single ordinary differential equation, and then the above governing equation was solved numerically. Effectiveness of lubrication approximation with various power law indices below 1 and various diverged angles was investigated in terms of normalized flow velocity and ratio of the approximate to exact flow rate. As the power law index decreases and diverged angle increases, the error of lubrication approximation becomes increasingly larger. It was shown that the proper selection of the constitutive law should be considered first to make the lubrication approximation work.

INTRODUCTION

The lubrication approximation has been an essential assumption in obtaining analytic solutions to the converging or diverging problems in polymer processings, particularly in both calendering and coating. Since this problem was first solved for Newtonian fluids long time ago, its effectiveness has been reconfirmed through the works done by M.M. Denn and S. Middleman[1,2]. Excellence of the analytic solution has never diminished notwithstanding the development of various numerical methods with the advance in digital computers, but also such solution for the simplified system could be used as one of the means to check the validity of the numerical solution for the more complicated situation. Thus, the purpose of this study is to evaluate the usefulness of the lubrication approximation by examining closely its error for the flow of non-Newtonian fluids in simple geometry. Specifically, for the pressure-driven flow of the power law fluid through a two-dimensional duct with nonparallel walls, effectiveness of the lubrication approximation will be investigated with various power law indices and various diverged angles. It is performed by comparing the exact numerical solution with the approximate solution obtained by using the lubrication approximation.

MATHEMATICAL DEVELOPMENT

Let us consider the problem of pressure-driven flow through a two-dimensional duct with nonparallel walls, as shown in Fig. 1. The total angle between the plates is 2α, with the walls at $\theta = \pm\alpha$. We take the pressure to be $p = \Delta P$ at $r = r_o$, $p = 0$ at $r = r_o + L$, and $k = (r_o + L)/r_o$. It is well known that the converging flow problem is exactly the same as the diverging case if the inertia term is negligible. Here we restrict ourselves to the diverging flow problem only. It is assumed the flow is entirely radial, in that fluid particles move from the vertex along lines of constant θ, so that $v_\theta = 0$. The continuity equation in polar coordinates is then

$$\frac{1}{r}\frac{\partial}{\partial r}(r\ V_r) = 0 \qquad (1)$$

That is, $r\ V_r =$ is independent of r, so equation (1) integrates to

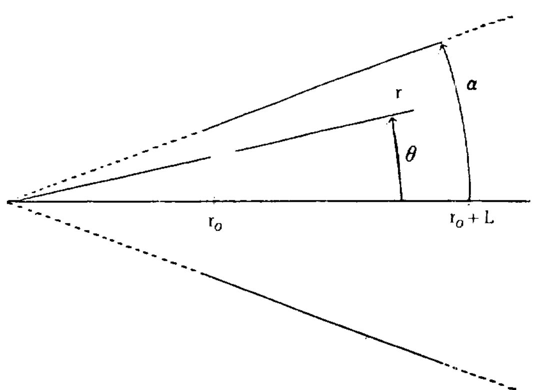

Fig. 1. A two-dimensional duct with nonparallel walls in polar coordinates.

$$V_r=\frac{f(\theta)}{r} \tag{2}$$

Some restrictions on the function $f(\theta)$ are imposed by boundary conditions. The velocity must vanish at the side walls, $\theta=\pm\alpha$, because of the no-slip condition, so the function $f(\theta)$ must vanish for $\theta=\pm\alpha$: $f(+\alpha)=f(-\alpha)=0$. The geometric symmetry makes $f(\theta)$ to be even, so the first derivative of $f(\theta)$ becomes zero at $\theta=0$. Furthermore as shown later in equation (18), it is convenient to specify $f(\theta)$ at $\theta=0$ instead of the flow rate per unit width Q for the numerical calculations.

The r and θ components of the dynamic equation, with steady, no inertia, and no gravity conditions, become

$$\frac{\partial T_{rr}}{\partial r}+\frac{1}{r}\frac{\partial T_{r\theta}}{\partial\theta}+\frac{1}{r}(T_{rr}-T_{\theta\theta})=0 \tag{3}$$

$$\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta}+\frac{\partial T_{r\theta}}{\partial r}+\frac{\partial}{\partial r}T_r=0 \tag{4}$$

and

$$T_{rr}=-p+2\mu\frac{\partial V_r}{\partial r} \tag{5}$$

$$T_{\theta\theta}=-p+2\mu\frac{V_r}{r} \tag{6}$$

$$T_{r\theta}=\mu\frac{1}{r}\frac{\partial V_r}{\partial\theta} \tag{7}$$

The details can be found in the book of P.C. Lu[3].

For the power law fluid, viscosity μ is a function of the second invariant II of the appropriate velocity gradient tensor.

$$\mu=K\left|\frac{1}{2}\,\mathrm{II}\right|^{(n-1)/2} \tag{8}$$

where K is the consistency factor, n the power law index, and

$$\frac{1}{2}\,\mathrm{II}=2\left\{\left(\frac{\partial V_r}{\partial r}\right)^2+\left(\frac{V_r}{r}\right)^2\right\}+\left(\frac{1}{r}\frac{\partial V_r}{\partial\theta}\right)^2$$
$$=(4f^2+f'^2)/r^4 \tag{9}$$

Therefore,

$$\mu=Kg(\theta)/r^{2(n-1)} \tag{10}$$

where

$$g(\theta)=(4f^2+f'^2)^{(n-1)/2} \tag{11}$$

Now, substituting equations (5) to (7) and (10) into equations (3) and (4) yields the following relations of functions $f(\theta)$ and $g(\theta)$.

$$r^{2n+1}\frac{\partial p}{\partial r}=K\{2(n-1)fg+f'g'+f''g\} \tag{12}$$

$$r^{2n}\frac{\partial p}{\partial\theta}=K\{2fg'-2(n-2)f'g\} \tag{13}$$

It is easy to show that

$$\frac{p}{K}=C_0+C_1(\theta)/r^{2n} \tag{14}$$

where C_0 is constant independent of θ,r and C_1 is a function of θ only. From equations (12) and (13),

$$-2nC_1=4(n-1)fg+f'g'+f''g \tag{15}$$

$$C_1'=2fg'-2(n-2)f'g \tag{16}$$

Finally, the governing equation of function $f(\theta)$ is derived from above equations

$$(f'g)''+4(3n-n^2)f'g+4(2n-1)fg'=0 \tag{17}$$

for $0\le\theta\le\alpha$. The boundary conditions are

$$f(0)=f_0 \tag{18}$$

$$f'(\theta)=0 \tag{19}$$

$$f(\alpha)=0 \tag{20}$$

Equation (17) is a non-linear 3rd order ordinary differential equation, which can be normally solved by the numerical means. Once $f(\theta)$ is solved, $f(\theta)$ and $C_1(\theta)$ give the exact flow rate and pressure difference per unit width.

$$Q_{ext}=2\int_0^\alpha rV_r d\theta=2\int_0^\alpha f(\theta)d\theta \tag{21}$$

$$\frac{\Delta P_{ext}}{K}=\frac{1}{r_o^{2n}}\left(1-\frac{1}{k^{2n}}\right)\frac{\tan^{2n}\alpha}{\alpha}\int_0^\alpha C_1(\theta)d\theta \tag{22}$$

On the other hand, the following lubrication approximation can be obtained easily.

$$\frac{\Delta P_{app}}{K}=\frac{Q_{app}^n}{\tan\alpha}\frac{1}{r_o^{2n}}\left(1-\frac{1}{k^{2n}}\right)\frac{1}{2n}\left(\frac{1+2n}{2n}\right)^n \tag{23}$$

The following equation is obtained from the condition to have the same ΔP in equations (22) and (23).

$$\frac{Q_{app}}{Q_{ext}}=(2n)^{1/n}\frac{n}{1+2n}\frac{(\tan\alpha)^{2+\frac{1}{n}}}{\alpha^{1/n}}\frac{\left(\int_0^\alpha C_1(\theta)d\theta\right)^{1/n}}{\int_0^\alpha f(\theta)d\theta} \tag{24}$$

Numerical strategy

It is not so difficult to solve numerically equation (17). To solve this equation which is a non-linear 3rd order ordinary differential equation, a combination of non-linear shooting and 4th order Runge-Kutta algorithms was developed, and a half interval method was applied to seek for the boundary condition $f''(0)$ wanted in the shooting algorithm. 2nd order Simpson's integration rule was also used in integration of $f(\theta)$ and $C_1(\theta)$.

RESULTS AND DISCUSSION

The effectiveness of the lubrication approximation on power law fluids flowing between nonparallel plates can be appreciated by observing the variation on the power law index n and the diverged angle α. It is useful for purposes of presentation to scale the dependent and independent variables by equations (25) and (26). A normalized angle ϕ, and a normalized flow velocity F are defined, as follows:

$$\phi = \frac{\theta}{\alpha} \tag{25}$$

$$F = \frac{\alpha f}{Q} \tag{26}$$

Fig. 2 shows the normalized velocity profile $F(\phi)$ for a appropriately selected $\alpha = 5°$ when the power law indices are 1, 0.7 and 0.4 respectively. It is seen that $F(\phi)$ becomes increasingly flatter as n decreases (This tendency comes to be clear from surplus numerical results.). Variation with the diverged α is shown in Fig. 3, where $F(\phi)$ becomes increasingly sharper as α increases. Though Fig. 3 is the result in case of $n = 0.7$, analogous results were also obtained under other n below 1. The limiting value $\lim_{\alpha \to 0} F(\phi) = \frac{3}{4}(1-\phi^2)$ for Newtonian flow shows exactly the same results obtained by the numerical calculations. The curves in Figs. 2 and 3 suggest that the diverged angle acting as the geometrical factor is relatively less sensitive than the power law index as the constitutive law.

Effectiveness of the lubrication approximation with n and α can be investigated by comparing flow rate Q_{app} obtained by the approximation with the exact flow rate Q_{ext} to give the same pressure difference. Fig. 4 shows plain variations of ratio Q_{app}/Q_{ext} with n and α. It is seen in Fig. 4 that Q_{app} deviates increasingly faster as n decreases and α increases, and furthermore the lubrication approximation for the fluid with small n would make a serious error even for the geometry with small angle α.

The severe error in lubrication approximation for small n despite of the moderate and gradual variation of $F(\phi)$ is due to the inverse exponent of n in equation (24). It thus appears that application of the lubrication approximation for small n rises to serious errors.

On the other hand, to see the geometric effect in lubrication approximation, the ratio Q_{app}/Q_{ext} in the case of the expanding tube for Newtonian fluids was plotted with dotted line in Fig. 4. This happens to near-

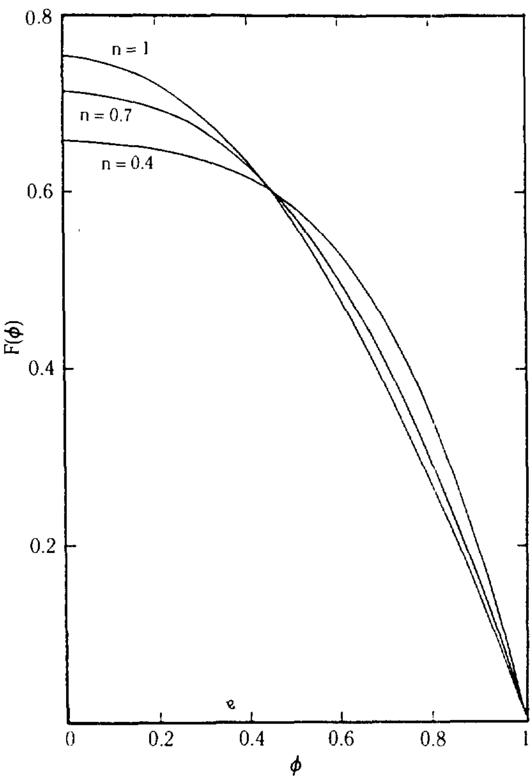

Fig. 2. Normalized flow velocity functions $F(\phi)$ for $\alpha = 5°$.

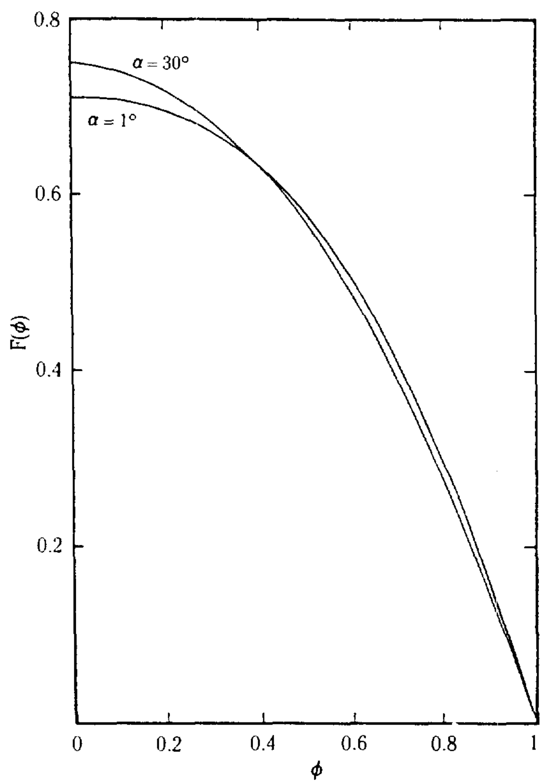

Fig. 3. Normalized flow velocity functions $F(\phi)$ for the fluid with $n = 0.7$.

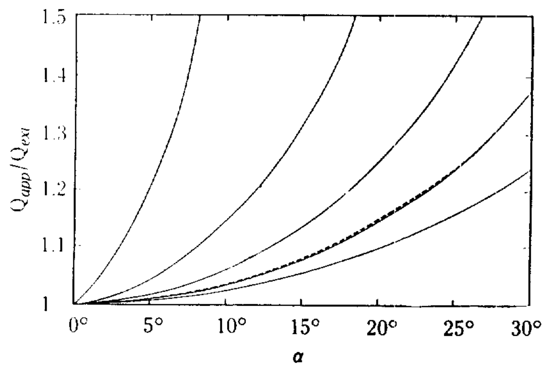

Fig. 4. Ratio of approximate to exact flow rate, Q_{app}/Q_{ext} against α.

The solid curves are for the power law fluids (from the bottom, n = 1, 0.8, 0.6, 0.4, and 0.2) flowing between nonparallel walls. The dotted curve is for the Newtonian fluid flowing through a expanding tube.

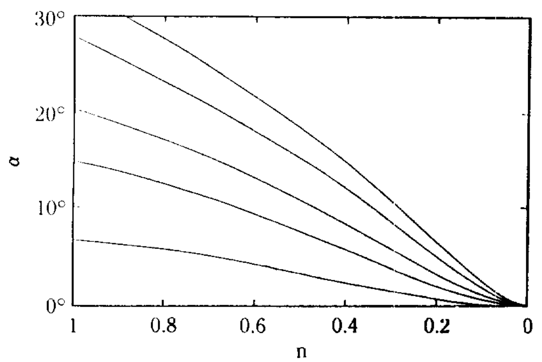

Fig. 5. The angle α to give the allowable % error in the lubrication approximation against the power law fluid index n (From the top, the allowable % error are 30%, 20%, 10%, 5%, and 1% respectively).

ly coincide with the curves of power law flow for n = 0.8 in plain geometry. The diverged angle to give % error in the lubrication approximation is shown in Fig. 5, which was reconstructed from the raw data shown in Fig. 4 using 4-node Lagrangian interpolation. Fig. 5 gives a clear-cut criterion on the lubrication approximation to be applied within an permissible error.

CONCLUSION

The lubricated flow problem for the power law fluid, so difficult to obtain the exact analytic solution, was successfully reduced to a single ordinary differential equation. This means that the solution can be obtained with little further effort. Though this study is restricted on the flow with no inertia in plain geometry, it is sufficient to investigate the virtue of lubrication approximation itself. As the power law index decreases and the diverged angle increases, their non-linearly combined effects on the lubrication approximation act increasingly. It was shown that the proper selection of the constitutive law should be considered first to make the lubrication approximation work.

NOMENCLATURE

c_0	:	a constant defined in equation (14)
$c_1(\theta)$	:	a function defined in equation (14)
$F(\phi)$	:	a normalized flow velocity defined by equation (26)
$f(\theta)$	:	a function defined by equation (2)
f_o	:	f at $\theta = 0$
$g(\theta)$	:	a function defined by equation (11)
K	:	the consistency factor defined in equation (7)
k	:	ratio of inlet to outlet section of plates
L	:	length of plates
n	:	the power law index defined in equation (7)
p	:	pressure of fluid
P	:	pressure at inlet section of plates
ΔP	:	pressure difference from inlet to outlet section of plates
Q	:	flow rate per unit width
r	:	a radial position in polar coordinate system
r_o	:	r at inlet section of plates as shown in Fig. 1
V_r, V_θ	:	the velocity components of r and θ
$T_{rr}, T_{r\theta}, T_{\theta\theta}$	:	the elements of stress tensor in the polar coordinate system defined by equations (5) to (7)
$'$, $''$	:	denote 1st and 2nd derivatives respectively

Greek Letters

α	:	the diverged angle of nonparallel plates
ϕ	:	a normalized angle by defined equation (25)
θ	:	an angle in polar coordinate system
μ	:	viscosity
II	:	a second invariant defined by equation (9)

Subscripts

ext	:	refers to the exact numerical solution
app	:	refers to the lubrication approximation

REFERENCES

1. Denn, M.M.: "Process Fluid Mechanics", Ch.10, Prentice-Hall, Inc. (1980).
2. Middleman, S.: "Fundamental of Polymer Processing", Ch.8, MacGraw Hill, Inc. (1977).
3. Lu, P.C.: "Introduction to the Mechanics of Viscous Fluid", Ch.8, Hemisphere Pub. Corp. (1977).